Fourier Transform (9)

Problem (16) Find $f(x)$, if its fourier cosine transform is $\frac{1}{1+s^2}$, Or, if $\overline{f_c}(s) = F_c\{f(x)\}$ then find $f(x)$.

Answer

$$\because f(x) = F_c^{-1}\{\overline{f_c}(s)\} = F_c^{-1}\left\{\frac{1}{1+s^2}\right\}.$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+s^2}\cdot \cos sx \, ds \quad \text{--- (1)}$$

$$\Rightarrow \frac{df(x)}{dx} = \frac{2}{\pi}\int_0^\infty \frac{-\sin sx \cdot s}{1+s^2} \, ds \quad \text{--- (2)}$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{s^2 \sin sx}{s(1+s^2)} ds$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{(1+s^2-1)}{s(1+s^2)} \cdot \sin sx \, ds.$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s} ds + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)} ds$$

$$= -\frac{2}{\pi}\cdot\frac{\pi}{2} + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)} ds.$$

$$\Rightarrow \frac{df}{dx} = -1 + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)} ds \quad \text{--- (3)}$$

$$\Rightarrow \frac{d^2f}{dx^2} = \frac{2}{\pi}\int_0^\infty \frac{s \cos sx}{s(1+s^2)} ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{1+s^2}\cdot \cos sx \, ds = f$$

$$\Rightarrow \frac{d^2f}{dx^2} - f = 0$$

Its complementry solution is

$$f = Ae^{-x} + Be^{x} \quad \text{--- (4)}$$

Putting $x = 0$ in (1), we have

$$f(0) = \frac{2}{\pi}\int_0^\infty \frac{ds}{1+s^2} = \frac{2}{\pi}\left[\tan^{-1}s\right]_0^\infty = 1$$

Putting $x = 0$ in (3), we have

$$\frac{df}{dx} = -1$$

$\therefore$ $f = 1$, when $x = 0$ --- (5)

$\frac{df}{dx} = -1$, when $x = 0$ --- (6)

Now by (4), we have

$A + B = 1$ [by (5) & (6)] --- (7)

Diffing (4) w. r. t. x

$$\frac{df}{dx} = -Ae^{-x} + Be^{x}$$

applying (5) & (6), we have

$-A + B = -1$ --- (8)

Solving (7) & (8), we have, $B = 0$ and $A = 1$

Now by (4), solution is

$f = e^{-x}$ Ans