$(D+1)^2 y = t$

$\Rightarrow (D^2+2D+1)\, y = t$

$\Rightarrow D^2y + 2Dy + y = t$

$\Rightarrow y'' + 2y' + y = t$

Taking Laplace transform of both side, we get

$\therefore L\{y''\} + 2L\{y'\} + L\{y\} = L\{t\}$

$\Rightarrow p^2 L\{y\} - p\,y(0) - y'(0) + 2[pL\{y\} - y(0)]$
$+ L\{y\} = \frac{L!}{p^2}$

$\Rightarrow (p^2+2p+1)L\{y\} - p\times(-3) - A - 2\times(-3) = \frac{1}{p^2}$

$\{\text{Let } A = y'(0)\}$

$\Rightarrow (p^2+2p+1)\, L\{y\} + 3p - A + 6 = \frac{1}{p^2}$

$\Rightarrow (p^2+2p+1)\, L\{y\} = \frac{1}{p^2} - 3p - 6 + A$

$\Rightarrow L\{y\} = \frac{1}{p^2(p^2+2p+1)} - 3\frac{p}{(p^2+2p+1)} - \frac{6}{p^2+2p+1}$
$+ \frac{A}{p^2+2p+1}$

$\Rightarrow L\{y\} = \frac{1}{p^2(p+1)^2} - 3\frac{p}{(p+1)^2} - \frac{6}{(p+1)^2} + \frac{A}{(p+1)^2}$

$\Rightarrow y = L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\} - 3L^{-1}\left\{\frac{(p+1)-1}{(p+1)^2}\right\} - 6L^{-1}\left\{\frac{1}{(p+1)^2}\right\}$
$+ A\,L^{-1}\left\{\frac{1}{(p+1)^2}\right\}$

$\Rightarrow y = L^{-1}\frac{1}{p^2(p+1)^2} - 3e^{-t}L^{-1}\left\{\frac{p-1}{p^2}\right\} - 6e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\}$
$+ A\,e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\}$

$$\Rightarrow y = L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\} - 3e^{-t}L^{-1}\left\{\frac{1}{p} - \frac{1}{p^2}\right\} - 6e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\}$$
$$+ Ae^{-t}L^{-1}\left\{\frac{1}{p^2}\right\}$$

$$\Rightarrow y = L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\} - 3e^{-t}\left(1 - \frac{t}{\lfloor 1}\right) - 6e^{-t}\frac{t}{\lfloor 1} + Ae^{-t}\cdot\frac{t}{\lfloor 1}$$

$$\Rightarrow y = L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\} - 3e^{-t} + 3te^{-t} - 6e^{-t}t + Ate^{-t}$$

$$\Rightarrow y = L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\} - 3e^{-t} - 3te^{-t} + Ate^{-t} \quad (1)$$

For $L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\}$

$$L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\} = L^{-1}\left\{\frac{1}{p^2}\cdot\frac{1}{(p+1)^2}\right\}$$
$$= L^{-1}\left\{f(p)\cdot g(p)\right\}$$

where
$$f(p) = \frac{1}{p^2}, \quad g(p) = \frac{1}{(p+1)^2}$$

$$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p^2}\right\} = \frac{t}{\lfloor 1} = t$$

and $L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{(p+1)^2}\right\} = e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\} = e^{-t}t$

Hence by convolution theorem, we have,

$$L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\} = \int_0^t x e^{-(t-x)}(t-x)\,dx$$

$$= e^{-t}\int_0^t x e^{x}(t-x)\,dx$$

$$= e^{-t}\int_0^t (xe^{x}t - x^2e^{x})\,dx$$

$$\therefore L^{-1}\left\{\frac{1}{p(p+1)^2}\right\} = te^{-t}\int_0^t xe^x\,dx - e^{-t}\int_0^t x^2e^x\,dx$$

$$= te^{-t}\left[xe^x - e^x\right]_0^t - e^{-t}\left[x^2e^x\right]_0^t + e^{-t}\int_0^t 2xe^x\,dx$$

$$= te^{-t}\left[te^t - e^t + 1\right] - e^{-t}\left[t^2e^t - 0\right] + e^{-t}2\left[xe^x - e^x\right]_0^t$$

$$= te^{-t}\left[te^t - e^t + 1\right] - t^2 + 2e^{-t}\left[te^t - e^t + 1\right]$$

$$= \cancel{t^2} - t + te^{-t} - \cancel{t^2} + 2t - 2 + 2e^{-t}$$

$$= 2e^{-t} + te^{-t} + t - 2$$

$\therefore$ (1) becomes

$$y = 2e^{-t} + te^{-t} + t - 2 - 3e^{-t} - 3te^{-t} + Ate^{-t}$$

$$y = -e^{-t} - 2te^{-t} + t - 2 + Ate^{-t} \quad \text{--- (2)}$$

Since $y = -1$ when $t = 1$

putting $t = 1$ in (2)

$$\therefore -1 = -e^{-1} - 2\times 1\times e^{-1} + 1 - 2 + A\times 1\times e^{-1}$$

$$\Rightarrow \cancel{-1} = -\frac{1}{e} - \frac{2}{e} \cancel{-1} + \frac{A}{e}$$

$$\Rightarrow \frac{A}{e} = \frac{3}{e}$$

$$\Rightarrow A = 3$$

$\therefore$ (2) becomes

$$y = -e^{-t} - 2te^{-t} + t - 2 + 3te^{-t}$$

$$\Rightarrow y = te^{-t} - e^{-t} + t - 2$$

Which is the required solution.

(7) Solve: $(D^2+1)y = t\cos 2t$, $y=0$, $\frac{dy}{dt}=0$ when $t=0$

Ans:

We have,

$$(D^2+1)y = t\cos 2t$$
$$\Rightarrow D^2y + y = t\cos 2t$$
$$\Rightarrow y'' + y = t\cos 2t$$

Taking laplace transform on both side, we get

$$\therefore L\{y''\} + L\{y\} = L\{t\cos 2t\}$$

$$\Rightarrow p^2 L\{y\} - p\,y(0) - y'(0) + L\{y\} = L\{t\cos 2t\}$$
$$\Rightarrow (p^2+1)L\{y\} - p\times 0 - 0 = L\{t\cos 2t\}$$
$$\Rightarrow (p^2+1)L\{y\} = L\{t\cos 2t\} \quad \text{--- (1)}$$

Now.

$$L\{\cos 2t\} = \frac{p}{p^2+4} = f(p) \quad \text{(say)}$$

$$\therefore L\{t\cos 2t\} = (-1)\frac{d}{dp}f(p)$$
$$= -\frac{d}{dp}\left\{\frac{p}{p^2+4}\right\}$$
$$= -\left[\frac{(p^2+4).1 - p(2p)}{(p^2+4)^2}\right]$$
$$= -\left[\frac{p^2+4-2p^2}{(p^2+4)^2}\right]$$
$$= \frac{p^2-4}{(p^2+4)^2}$$

$\therefore$ From (1),

$$(p^2+1)L\{y\} = \frac{p^2-4}{(p^2+4)^2}$$

$$\Rightarrow L\{y\} = \frac{(p^2-4)}{(p^2+4)^2(p^2+1)}$$

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$$\Rightarrow y = L^{-1}\left\{\frac{p^2-4}{(p^2+4)^2(p^2+1)}\right\} \quad \text{——— (2)}$$

Let $\dfrac{p^2-4}{(p^2+4)^2(p^2+1)} = \dfrac{A}{p^2+4} + \dfrac{B}{(p^2+4)^2} + \dfrac{Cp+D}{(p^2+1)}$ ——— (3)

$$\Rightarrow p^2-4 = A(p^2+4)(p^2+1) + B(p^2+1) + (Cp+D)(p^2+4)^2$$

$$\Rightarrow p^2-4 = A(p^4+p^2+4p^2+4) + B(p^2+1) + (Cp+D)(p^4+8p^2+16)$$

$$\Rightarrow p^2-4 = A(p^4+5p^2+4) + B(p^2+1) + (Cp^5+8Cp^3+16Cp+Dp^4+8Dp^2+16D)$$

$$\Rightarrow p^2-4 = Cp^5 + (A+D)p^4 + 8Cp^3 + (5A+B+8D)p^2 + 16Cp + 4A+B+16D$$

Equating the like term

$C = 0$ ——— (4)

$A+D = 0$ ——— (5)

$8C = 0$ ——— (6)

$5A+B+8D = 1$ ——— (7)

$16C = 0$ ——— (8)

$4A+B+16D = -4$ ——— (9)

From (5)

$A = -D$ ——— (10)

From (7) and (9)

$-5D+B+8D = 1$ {Using (5)}

and $-4D+B+16D = -4$

$\Rightarrow$ $B+3D = 1$ ——— (11)

and $B+12D = -4$ ——— (12)

From (11) and (12)

$-9D = 5 \Rightarrow D = -\dfrac{5}{9}$

Hence by Convolution theorem, we have

$$L^{-1}\left\{\frac{1}{(p^2+4)}\frac{1}{(p^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2x \, \frac{1}{2}\sin 2(t-x)\,dx$$

$$= \frac{1}{8}\int_0^t 2\sin 2x \cdot \sin(2t-2x)\,dx$$

$$= \frac{1}{8}\int_0^t \{\cos(2x-2t+2x) - \cos(2x+2t-2x)$$

$$= \frac{1}{8}\int_0^t \{\cos(4x-2t) - \cos 2t\}\,dx$$

$$= \frac{1}{8}\left[\frac{\sin(4x-2t)}{4}\right]_0^t - \frac{\cos 2t}{8}\left[x\right]_0^t$$

$$= \frac{1}{32}\left[\sin 2t - \sin(-2t)\right] - \frac{\cos 2t}{8}\left[t-0\right]$$

$$= \frac{2\sin 2t}{32} - \frac{t\cos 2t}{8}$$

$$= \frac{\sin 2t}{16} - \frac{t\cos 2t}{8}$$

∴ (13) becomes

$$y = \frac{5}{9}\times\frac{1}{2}\sin 2t + \frac{8}{3}\left[\frac{\sin 2t}{16} - \frac{t\cos 2t}{8}\right] - \frac{5}{9}\sin t$$

$$= \frac{5}{18}\sin 2t + \frac{\sin 2t}{6} - \frac{t\cos 2t}{3} - \frac{5}{9}\sin t$$

$$= \frac{(5+3)}{18}\sin 2t - \frac{t}{3}\cos 2t - \frac{5}{9}\sin t$$

$$= \frac{8}{18}\sin 2t - \frac{t}{3}\cos 2t - \frac{5}{9}\sin t$$

$$y = \frac{4}{9}\sin 2t - \frac{t}{3}\cos 2t - \frac{5}{9}\sin t$$

which is the required soln.

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