Partial Differential Equation with constant coefficients

A partial differential equation, which is linear with respect to dependent variable and its partial derivative and the coefficient are not function of independent variable, but merely constant is called Linear Partial differential equation with constant coefficient.

ie in the form

$$\left(\frac{\partial^n z}{\partial x^n} + A_1\frac{\partial^n z}{\partial y \partial x^{n-1}} + A_2\frac{\partial^n z}{\partial y^2 \partial x^{n-2}} + \cdots + A_n\frac{\partial^n z}{\partial y^n}\right)$$

$$+ \left(B_0\frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1\frac{\partial^{n-1} z}{\partial y \partial x^{n-2}} + \cdots + B_{n-1}\frac{\partial^{n-1} z}{\partial y^{n-1}}\right)$$

$$+ \cdots \quad \cdots + \left(M\frac{\partial z}{\partial x} + N\frac{\partial z}{\partial y}\right) + Bz = f(x,y)$$

Where $A_1, A_2, \cdots A_n, B_0, B_1 \cdots B_{n-1}, \cdots M, N, B$ are all constants.

Homogeneous Linear Partial Differential equation

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + A_1\frac{\partial^n z}{\partial y \partial x^{n-1}} + A_2\frac{\partial^n z}{\partial y^2 \partial x^{n-2}} + \cdots + A_n\frac{\partial^n z}{\partial y^n} = f(x,y)$$

where $A_1, A_2 \cdots A_n$ are constants is called Linear homogeneous Partial Differential Equation with constant coefficients. We can write

$$\left(D^n + A_1 D^{n-1} \cdot D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n\right) z = f(x,y)$$

ie $\phi(D, D')z = f(x,y)$

where $D \equiv \frac{\partial}{\partial x}$ & $D' \equiv \frac{\partial}{\partial y}$

Solution of Homogeneous Linear partial differential equation

Let $\phi(D, D')z = f(x, y)$ ——— (1)

be the given differential eqn, then its complete solution consist of two parts, (a) Complementary function (b) Particular Integral.

The complementary function is solution of

$$\phi(D, D')z = 0 \quad \text{——— (2)}$$

and particular integral is $\frac{1}{\phi(D, D')} f(x, y)$ ——— (3)

NOTE:- If RHS of eqn (1) is zero, ie $f(x, y) = 0$ then complete solution has no particular integral. and complete solution is z = complementary function.

To find Complementary Function

If given differential equation is

$$\phi(D, D')z = f(x, y) \quad \text{——— (1)}$$

Auxiliary equation is

$$\phi(m, 1) = 0 \quad \text{——— (2)}$$

{It is obtained by putting $D = m$ & $D' = 1$ in $\phi(D, D')$}

Case I

If $m = m_1, m_2, m_3 \cdots m_n$ be the n-distinct roots of auxiliary equation (2) then complementary function of (1) is

$$CF = \psi_1(y + m_1 x) + \psi_2(y + m_2 x) + \dots + \psi_n(y + m_n x)$$

Case II. If auxiliary equation ② has repeated roots.

Let $m = m_0$ is repeated r times, then contribution of this repeated root in Complementary function will be

$$f_1(y+m_0x) + xf_2(y+m_0x) + \cdots + x^{r-1}f_r(y+m_0x).$$

Examples

① Solve $2r + 5s + 2t = 0$

Solⁿ Given differential equation can be written as

$$(2D^2 + 5DD' + 2D')z = 0 \quad \text{——— ①}$$

Auxiliary equation is

$$2m^2 + 5m + 2 = 0$$

$$\Rightarrow 2m^2 + 4m + m + 2 = 0$$

$$\Rightarrow 2m(m+2) + 1(m+2) = 0$$

$$\Rightarrow (m+2)(2m+1) = 0$$

$$\Rightarrow m = -2, -\tfrac{1}{2}$$

Hence complete solution is

$z = CF$. (As it has no PI because its rhs is zero)

ie $z = \phi(y-2x) + f(y - \tfrac{1}{2}x)$

ie $z = \phi(y-2x) + \psi(2y-x)$

② Solve: $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 2\dfrac{\partial^3 z}{\partial x \partial y^2} = 0$

Solⁿ The given differential equation can be written as $(D^3 - 3D^2 \cdot D' + 2DD'^2)z = 0$ ——— ①

Auxiliary equation is

$$m^3 - 3m^2 + 2m = 0$$

$$\Rightarrow m(m^2-3m+2)=0$$

$$\Rightarrow m(m^2-2m-m+2)=0$$

$$\Rightarrow m\{m(m-2)-1(m-2)\}=0$$

$$\Rightarrow m(m-1)(m-2)=0$$

Roots of auxiliary eqns are

$$m=0, 1, 2.$$

Complete solution of ① is

$$Z = f_1(y+0.x) + f_2(y+1.x) + f_3(y+2.x)$$

ie $z = f_1(y) + f_2(y+x) + f_3(y+2x)$.

③ Solve : $\dfrac{\partial^4 z}{\partial x^4} - \dfrac{\partial^4 z}{\partial y^4} = 0$

Solⁿ The given equation can be written as

$$(D^4 - D'^4)z = 0 \quad \text{———} ①$$

Auxiliary equation is

$$m^4 - 1 = 0$$

$$\Rightarrow (m-1)(m+1)(m^2+1) = 0$$

$$\Rightarrow m = 1, -1, +i, -i \text{ are the roots.}$$

Hence complete solⁿ is

$$z = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix)$$

For Practice

① Solve : $r = a^2 t$

② Solve : $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

③ Solve : $25r - 40s + 16t = 0$

④ Solve : $(D^3 - 4D^2D' + 4D.D'^2)z = 0$

⑤ Solve : $(D^4 - 2D^3D' + 2D.D'^3 - D'^4)z = 0$