K.e.c

Fourier transform (10)

Problem (17) Find the Fourier Sine and Cosine transform of the function x^{m-1}.

Answer: Let $f(x) = x^{m-1}$, $F_c\{f(x)\} = \bar{f}_c(s)$ and $F_s\{f(x)\} = \bar{f}_s(s)$.

Now, $$\bar{f}_c(s) = \int_0^\infty f(x)\cos sx\, dx = \int_0^\infty x^{m-1}\cos(sx)\, dx \quad \text{---(1)}$$

and $$\bar{f}_s(s) = \int_0^\infty f(x)\sin sx\, dx = \int_0^\infty x^{m-1}\sin(sx)\, dx \quad \text{---(2)}$$

By definition of Gamma function

$$\Gamma(m) = \int_0^\infty e^{-x} \cdot x^{m-1}\, dx$$

Putting $x = isy$ and noting $i = e^{i\pi/2}$

Now we get $\Gamma(m) = \int_0^\infty e^{-isy}\,(isy)^{m-1}\,(is\,dy)$

$$\Rightarrow \Gamma(m) = \int_0^\infty e^{-isy}\,(is)^m\, dy$$

$$\Rightarrow \frac{\Gamma(m)}{s^m} = \int_0^\infty e^{-isy} \cdot y^{m-1} \cdot i^m\, dy$$

$$= \left(e^{\frac{i\pi}{2}}\right)^m \cdot \int_0^\infty e^{-isy}\, y^{m-1}\, dy$$

$$= \left(e^{\frac{i\pi}{2}}\right)^m \int_0^\infty e^{-isx} \cdot x^{m-1}\, dx$$

$$\Rightarrow \int_0^\infty e^{-isx} \cdot x^{m-1}\, dx = \frac{\Gamma(m)}{s^m} \cdot e^{-\frac{im\pi}{2}}$$

$$\Rightarrow \int_0^\infty (\cos sx - i\sin sx)\, x^{m-1}\, dx = \frac{\Gamma(m)}{s^m}\left\{\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right\}$$

Equating real and imaginary parts we have

$$\int_0^\infty x^{m-1}\cos(sx)\, dx = \frac{\Gamma(m)}{s^m}\cos\left(\frac{m\pi}{2}\right)$$

and $$\int_0^\infty x^{m-1}\sin(sx)\, dx = \frac{\Gamma(m)}{s^m}\sin\left(\frac{m\pi}{2}\right)$$

$$\Rightarrow \bar{f}_s(s) = \frac{\Gamma(m)}{s^m}\sin\left(\frac{m\pi}{2}\right)$$

and $$\bar{f}_c(s) = \frac{\Gamma(m)}{s^m}\cos\left(\frac{m\pi}{2}\right)$$

Answer.

Problem 18 Find the sine and cosine transforms of $\dfrac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}$.

Answer Let $f(x) = \dfrac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}$

(i) We have to find $F_s\{f(x)\} = \bar{f}_s(s)$.

$$\bar{f}_s(s) = \int_0^\infty f(x)\sin sx\, dx$$

$$= \int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}\cdot \sin(sx)\, dx$$

$$= \int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}\cdot \frac{e^{isx}-e^{-isx}}{2i}\, dx$$

$$= \frac{1}{2i}\int_0^\infty \left[\frac{e^{(a+is)x}-e^{(a-is)x}+e^{-(a-is)x}-e^{-(a+is)x}}{e^{\pi x}-e^{-\pi x}}\right] dx$$

Scanned with CamScanner

$$= \frac{1}{2i}\int_0^{\infty}\left[\frac{e^{(a+is)x} - e^{-(a+is)x}}{e^{\pi x} - e^{-\pi x}} - \frac{e^{(a-is)x} - e^{-(a-is)x}}{e^{\pi x} - e^{-\pi x}}\right]dx$$

$$= \frac{1}{2i}\left[\frac{1}{2}\tan\left(\frac{a+is}{2}\right) - \frac{1}{2}\tan\left(\frac{a-is}{2}\right)\right]$$ {from definite integral}

$$= \frac{1}{4i}\left[\frac{\sin\left(\frac{a+is}{2}\right)}{\cos\left(\frac{a+is}{2}\right)} - \frac{\sin\left(\frac{a-is}{2}\right)}{\cos\left(\frac{a-is}{2}\right)}\right]$$

$$= \frac{1}{4i}\left[\frac{\sin\left(\frac{a+is}{2}\right)\cos\left(\frac{a-is}{2}\right) - \cos\left(\frac{a+is}{2}\right)\sin\left(\frac{a-is}{2}\right)}{\cos\left(\frac{a+is}{2}\right)\cos\left(\frac{a-is}{2}\right)}\right]$$

$$= \frac{1}{2i}\left[\frac{\sin\left(\frac{a+is}{2} - \frac{a-is}{2}\right)}{2\cos\left(\frac{a+is}{2}\right)\cos\left(\frac{a-is}{2}\right)}\right]$$

$$= \frac{1}{2i}\left[\frac{\sin(is)}{\cos a + \cos(is)}\right] = \frac{\sinh(s)}{2\{\cos a + \cosh(s)\}}$$

$$= \frac{e^{s} - e^{-s}}{2\left(\cos a + \frac{e^{s} + e^{-s}}{2}\right)}$$

$$= \frac{e^{s} - e^{-s}}{2(2\cos a + e^{s} + e^{-s})}$$

Ans

Scanned with CamScanner

(ii) we have to find $F_c\{f(x)\} = \bar{f}_c(s)$.

Now $\bar{f}_c(s) = \int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}} \cdot \cos sx \, dx$

$$= \int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}} \cdot \frac{e^{isx}+e^{-isx}}{2} \, dx$$

$$= \frac{1}{2}\int_0^\infty \left[\frac{e^{(a+is)x}+e^{-(a+is)x}}{e^{\pi x}-e^{-\pi x}} + \frac{e^{(a-is)x}+e^{-(a-is)x}}{e^{\pi x}-e^{-\pi x}} \right] dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\sec\left(\frac{a+is}{2}\right) + \frac{1}{2}\sec\left(\frac{a-is}{2}\right)\right]$$

$$= \frac{1}{4} \cdot \frac{\cos\left(\frac{a-is}{2}\right) + \cos\left(\frac{a+is}{2}\right)}{\cos\left(\frac{a+is}{2}\right)\cos\left(\frac{a-is}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{2\cos\left(\frac{a}{2}\right)\cos\left(\frac{is}{2}\right)}{\cos a + \cos(is)} = \frac{1}{2} \cdot \frac{\cos\left(\frac{a}{2}\right)\cosh\left(\frac{s}{2}\right)}{\cos a + \cosh(s)}$$

$$= \frac{\cos\left(\frac{a}{2}\right) \cdot e^{\frac{s}{2}} + e^{-\frac{s}{2}}}{2\cos a + (e^{s} + e^{-s})}$$

Ans.