Fourier transform :

The integral transform of a function $f(x)$ denoted by $I[f(x)]$ is defined by

$$I[F(x)] = f(s) = F\{f(x)\} = \int_{x_1}^{x_2} f(x)\, K(s,x)\, dx$$

where $K(s,x)$ is called the kernal of the transformation and is a known function of s and x.

The $f(x)$ is ~~a known~~ said the inverse transformation of $f(s)$.

i) when $K(s,x) = e^{-sx}$, it leads to the laplace transform of $f(x)$. i.e.

$$L\{f(x)\} = f(s) = \int_0^{\infty} f(x)\, e^{-sx}\, dx$$

ii) where $K(s,x) = e^{isx}$, we have the fourier transform of $f(x)$ i.e.

$$F\{f(x)\} = f(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

The infinite fourier transform of $f(x)$:-

The infinite Fourier transform of $F(x)$, in $-\infty < x < \infty$, is denoted by $f(s)$ or $F\{f(x)\}$

(2)

and is defined as

$$f(s) = F\{F(x)\} = \int_{-\infty}^{\infty} F(x).e^{-isx}\,dx$$

The inverse formula for Fourier transform is given by

$$F(x) = F^{-1}\{f(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,e^{isx}\,ds$$

or

$$\boxed{\begin{aligned} \text{if } f(s) &= F\{F(x)\} = \int_{-\infty}^{\infty} F(x)\,e^{isx}\,dx \\ F(x) &= F^{-1}\{f(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,e^{-isx}\,ds \end{aligned}}$$

Fourier sine and cosine transforms:

1) The Fourier sine transform of $F(x)$, $0 < x < \infty$ is denoted by $f_s(s)$ or $F_s\{f(x)\}$ and defined by

$$\boxed{f_s(s) = F_s\{F(x)\} = \int_{0}^{\infty} F(x)\sin sx\,dx}$$

The inverse formula for Fourier sine transform is given by

$$\boxed{F(x) = F_s^{-1}\{f_s(s)\} = \frac{2}{\pi}\int_{0} f_s(s)\sin sx\,ds}$$

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(2) The Fourier cosine transform of $F(x)$ $0 < x < \infty$ is denoted by $F_c\{F(x)\}$ or $f_c(s)$. and

$$\boxed{f_c(s) = F_c\{F(x)\} = \int_0^\infty F(x)\cos sx\,dx}$$

The inverse formula for infinite fourier cosine transform is given by

$$\boxed{F(x) = F_c^{-1}\{f_c(s)\} = \frac{2}{\pi}\int_0^\infty f_c(s)\cos sx\,dx}$$

Finite fourier sine and cosine transform :

1) The finite fourier sine transform of $F(x)$ in $0 < x < c$, is defined as

$$\boxed{f_s(s) = \int_0^c F(x)\sin\left(\frac{n\pi x}{c}\right).\,dx}$$

where n is an integer.

The function ~~F(x)~~ $F(x)$ is called the inverse finite fourier sine transform of $f_s(x)$ which is given by

$$\boxed{F(x) = \frac{2}{c}\sum_{n=1}^{\infty} f_s(x)\sin\left(\frac{n\pi x}{c}\right)}$$

2) The finite Fourier cosine transform of $F(x)$ in $0 < x < c$ is defined as

$$\boxed{F_c(x) = \int_0^c F(x) \cos \frac{n\pi x}{c} \, dx}$$

The function $F(x)$ is called the inverse finite Fourier cosine transform of $F_c(x)$ which is given by

$$\boxed{F(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^{\infty} F_c(x) \cos \frac{n\pi x}{c}}$$

Property (Theorem)

① Linear property: If a & b are constants then prove that

$$F\{aF(x) \pm bG(x)\} = aF\{F(x)\} \pm bF\{G(x)\}.$$

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$$F\{aF(x) + bG(x)\} = \int_{-\infty}^{\infty} e^{-isx} \{aF(x) \pm bG(x)\} dx$$

$$= \int_{-\infty}^{\infty} a e^{-isx} F(x) \, dx \pm \int_{-\infty}^{\infty} b e^{-isx} G(x) \, dx$$

$$= aF\{F(x)\} \pm bF\{G(x)\}$$