Particular Integral. (P.I.)

If differential equation is $f(D,D')z = \phi(x,y)$ —— ①

We know that the general value of z is sum of complementary function & particular integral.

We have already discussed the methods of finding complementary functions.

Now we will study, how to find particular integral?

Particular integral of diff eqn ① can be written as $\dfrac{1}{f(D,D')}\phi(x,y)$

If $\phi(x,y)$ is polynomial in x & y then we can find P.I. by expanding $f(D,D')$ as discussed in following examples.

Examples

① Solve $(D^2-2DD'+D'^2)z = 12xy$.

Sol^n Auxiliary eqn is

$$m^2-2m+1=0$$
$$\Rightarrow (m-1)^2=0$$
$$\Rightarrow m=1,1$$

So complementary function (CF) is given by

$$CF = f_1(y+x)+x f_2(y+x)$$

Now $P.I = \frac{1}{D^2 - 2DD' + D'^2} 12xy$

$$= \frac{1}{(D-D')^2} 12xy$$

$$= \frac{1}{D^2} \frac{1}{\left(1 - \frac{D'}{D}\right)^2} 12xy$$

$$= \frac{1}{D^2} \left(1 - \frac{D'}{D}\right)^{-2} 12xy$$

$$= \frac{1}{D^2} \left(1 + \frac{2D'}{D} + \frac{3D'^2}{D^2} + \cdots\right) 12xy$$

$$= \left(\frac{1}{D^2} + \frac{2D'}{D^3} + \frac{3D'^2}{D^4} + \cdots\right) 12xy$$

$$= \frac{1}{D^2}(12xy) + \frac{2}{D^3}(12x) + 0 + \cdots$$

($\because D'$ means partial derivative w.r.to y)

$$= 12 \cdot \frac{x^3}{6} y + 2 \times 12 \times \frac{x^4}{24}$$

($\because \frac{1}{D}$ mean integration w.r.to x.)

$$= 2x^3y + x^4.$$

Hence complete solution of given partial differential eqn is

$$z = CF + PI.$$

$$\Rightarrow z = f_1(y+x) + x f_2(y+x) + 2x^3y + x^4.$$

(2) Solve $(D^2+3DD'+2D'^2)z = x+y$

Sol^n Here auxiliary equation is

$$m^2+3m+2=0$$
$$\Rightarrow m^2+2m+m+2=0$$
$$\Rightarrow m(m+2)+1.(m+2)=0$$
$$\Rightarrow (m+1)(m+2)=0$$
$$\Rightarrow m=-1,-2.$$

$$\Rightarrow CF = f_1(y-x)+f_2(y-2x)$$

Now, $$P.I. = \frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$= \frac{1}{D^2}\frac{1}{\left(1+3\frac{D'}{D}+\frac{2D'^2}{D^2}\right)}(x+y).$$

$$= \frac{1}{D^2}\left(1+\frac{3D'}{D}+\frac{2D'^2}{D^2}+\cdots\right)^{-1}(x+y)$$

$$= \frac{1}{D^2}\left(1-\frac{3D'}{D}-\frac{2D'^2}{D^2}+\cdots\right)(x+y)$$

$$= \left(\frac{1}{D^2}-\frac{3D'}{D^3}+\cdots\right)(x+y)$$

$$= \frac{1}{D^2}(x+y)-\frac{3}{D^3}(1)$$

$$= \frac{x^3}{6}+\frac{x^2y}{2}-\frac{3x^3}{6}$$

$$= -\frac{1}{3}x^3+\frac{x^2y}{2}$$

Complete solution is

$$Z = CF + PI$$

ie $$z = f_1(y-x)+f_2(y-2x)-\frac{1}{3}x^3+\frac{x^2y}{2}$$

Ans