FOURIER TRANSFORM (FINITE) (II)

Definition ① The Finite Fourier Sine transform of $F(x)$:
The finite fourier Sine transform of $F(x)$, where $0<x<l$, is defined by

$$F_s\{F(x)\} = f_s(s) = \int_0^l F(x) \sin\frac{s\pi x}{l}\, dx,$$

where s is a positive integer. The function $F(x)$ is then called the inverse finite Fourier Sine transform of $f_s(s)$ and is given by

$$F_s^{-1}\{f_s(s)\} = F(x) = \frac{2}{l}\sum_{s=1}^{\infty} f_s(s) \sin\frac{s\pi x}{l}.$$

Definition ② The Finite Fourier Cosine transform of $F(x)$:
The finite fourier Cosine transform of $F(x)$, where $0<x<l$, is defined by

$$F_c\{F(x)\} = f_c(s) = \int_0^l F(x) \cos\frac{s\pi x}{l} \cdot dx,$$

where s is a positive or zero integer. The function $F(x)$ is then called the inverse finite cosine transform of $f_c(s)$ and is given by

$$F_c^{-1}\{f_c(s)\} = F(x) = \frac{1}{l} f_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} f_c(s) \cos\frac{s\pi x}{l}.$$

Theorem ① Fourier integral Theorem

Statement: If $f(x)$ satisfies the following conditions
(i) $f(x)$ satisfies the Dirichlet Conditions in every interval $-l\leq x\leq l$. (ii) $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges ie $f(x)$ is absolutely integrable in the interval $-\infty < x < \infty$, then

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$$f(x) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\int_{t=-\infty}^{\infty} f(t) \cos s(x-t)\, ds\, dt$$

The integral on R.H.S. is called Fourier integral or Fourier integral expansion of $f(x)$.

Proof: Let us consider a function $f(x)$ satisfying Dirichlet's condition in every interval $(-c, c)$, however large. Also let $\int_{-\infty}^{\infty} |f(x)|\,dx$ is convergent. (1) Then in the interval $(-c, c)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right) \quad (2)$$

where $a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\,dt$, $n = 0, 1, 2, \ldots$

and $b_n = \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\,dt$, $n = 1, 2, 3, \ldots$

Putting the values of a_n and b_n in (2),

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c} f(t)\left\{\cos\frac{n\pi t}{c}\cos\frac{n\pi x}{c} + \sin\frac{n\pi t}{c}\sin\frac{n\pi x}{c}\right\}dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c} f(t)\cos\frac{n\pi}{c}(t-x)\,dt.$$

Making use of the fact that $f(x)$ is uniformly convergent in the closed interval $-c \le x \le c$, then we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\int_{-c}^{c} f(t)\left[\sum_{n=1}^{\infty}\cos\frac{n\pi(t-x)}{c}\,dt\right]$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\left[1 + \lim_{n\to\infty}\sum_{r=1}^{n} 2\cos\frac{r\pi(t-x)}{c}\right]dt$$

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$$f(x) = \frac{1}{2c}\int_{-c}^{c}\left[1 + \lim_{n\to\infty}\sum_{r=1}^{n}\left\{\cos\frac{r\pi(t-x)}{c} + \cos\frac{-r\pi(t-x)}{c}\right\}\right]f(t)\,dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\left[1 + \lim_{n\to\infty}\sum_{r=-n}^{n}\cos\frac{r\pi(t-x)}{c}\right]dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\lim_{n\to\infty}\sum_{r=-n}^{n}\frac{1}{c/\pi}\cos\frac{r(t-x)}{c/\pi}\right]dt$$

Making use of definition of integral as a limit of sum, we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{2\pi}\int_{-c}^{c}\left[\int_{-\infty}^{\infty}\cos u(t-x)\,du\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{2\pi}\int_{-c}^{c} f(t)\,dt\int_{-\infty}^{\infty}\cos u(t-x)\,du$$

Making $c \to \infty$ and using (1)

$$f(x) = 0 + \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty}\cos u(t-x)\,du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\,dt\,du$$

Finally if $-\infty < x < \infty$, then

$$f(x) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\int_{t=-\infty}^{\infty} f(t)\cos s(t-x)\,ds\,dt$$

Proved