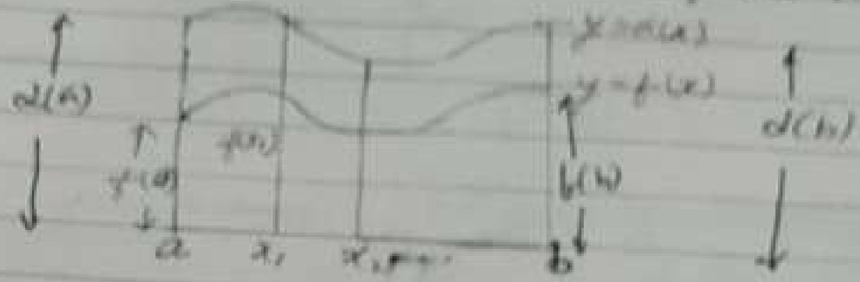
Integral Transform

<u>Note:</u>

If $y = f(x)$, $x \in [a, b]$ is the curve defined in $[a, b]$ and $y = \alpha(x)$ is another curve defined in $[a, b]$.



<u>Bounds of a set :-</u>

S = a set

If, $\exists$ two finite numbers b and B such that

$$b < x < B, \ \forall \ x \in S$$

we say S is bounded

B = an upper bound of S

b = a lower bound of S

Once an upper bound of S exists, we will have infinity of upper bounds to S.

The least of these upper bounds of S is called least upper bound (l.u.b) or sup S or simply the $\overline{bd}$ of S or the exact upper bound of S.

Similarly once a lower bound of S exists, it will have infinity of lower bounds and greatest of these lower bounds is called greatest lower bound (g.l.b) of S or inf S or the $\underline{bd}$ of S or exact lower bound of S.

A set S, having upper and lower bound both is called a bounded set

Properties of Sup(S):-

$Sup(S) = \lambda$ (say)

$x \le \lambda$, $\forall\ x \in S$

$\Rightarrow$ (i) $x < (\lambda+\epsilon)$, $\forall\ \epsilon>0$ $\forall\ x \in S$

(ii) $\exists$ at least one member say x_1 in S such that
$\lambda - \epsilon < x_1$

Properties of inf(S):-

$\inf(S) = \mu$ (say)

$x \geq \mu$, $\forall\ x \in S$

$\Rightarrow$ (i) $\mu - \epsilon < x$, $\forall\ x \in S$, $\epsilon > 0$

(ii) $\exists$ at least one member say y_1 in S such that
$y_1 < \mu + \epsilon$

Note:- A function $f(x)$ defined in $[a,b]$ is bounded

$\Rightarrow$ $\mu \le f(x) \le \lambda$, $\forall\ x \in [a,b]$

$\Rightarrow$ $f(x)$ is bounded in each sub-interval of $[a,b]$

VVI Q. Define Stieltje's integral [or Riemann-Stieltje's integral] of $f(x)$ w.r.t. $\alpha(x)$ from a to b and how the definition is extended to include complex function. [R.U. 72,74]

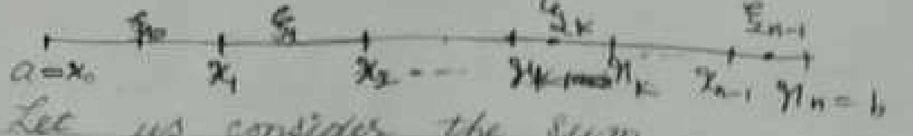
Ans

Let $f(x)$ and $\alpha(x)$ be real functions of the real variable x and these are defined in $[a,b]$ i.e. $a \le x \le b$

Let Δ = the partition of $[a,b]$ by arbitrary set of points say $a = x_0, x_1, x_2, \ldots, x_K, \ldots, x_n = b$

where $x_0 < x_1 < x_2 < \ldots < x_n$

Take the points $\xi_0, \xi_1, \xi_2, \ldots \xi_{n-1}$ in the sub intervals so obtained by the partition Δ of $[a,b]$, where $\xi_K \in [x_K, x_{K+1}]$, $K = 0, 1, 2, \ldots, (n-1)$



Let us consider the sum

$$f(\xi_0)[\alpha(x_1)-\alpha(x_0)] + f(\xi_1)[\alpha(x_2)-\alpha(x_1)] + \cdots$$

$$\cdots + f(\xi_{n-1})[\alpha(x_n)-\alpha(x_{n-1})]$$

$$= \sum_{k=0}^{n-1} f(\xi_k)[\alpha(x_{k+1})-\alpha(x_k)]$$

Let we increase the number of points in the partition indefinitely such that greatest of the length like $(x_{k+1}-x_k)$, called norm of the partition Δ, denoted by δ tends to zero.

If $\lim\limits_{\delta\to 0} \sum\limits_{k=0}^{n-1} f(\xi_k)[\alpha(x_{k+1})-\alpha(x_k)]$ exists.

i.e. finite and definite, whatever be the character of Δ and the choice of ξ_k in the sub-intervals $I_k \equiv [x_k, x_{k+1}]$, then this limit is called the Stieltjes integral of $f(x)$ w.r.t $\alpha(x)$ from a to b & is denoted by $\int_a^b f(x)\,d\alpha(x)$.

Thus, $$\int_a^b f(x)\,d\alpha(x) = \lim_{\delta\to 0} \sum_{k=0}^{n-1} f(\xi_k)[\alpha(x_{k+1})-\alpha(x_k)]$$

<u>nd Part:-</u>

Let $f(x)$ and $\alpha(x)$ be complex functions, defined by $f(x) = f_1(x) + i f_2(x)$, where $f_1(x)$, $f_2(x)$ are real functions

and $\alpha(x) = \alpha_1(x) + i\,\alpha_2(x)$, where $\alpha_1(x)$, $\alpha_2(x)$ are real functions, then

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b [f_1(x) + i f_2(x)][d\alpha_1(x) + i\,d\alpha_2(x)]$$

$$= \int_a^b f_1(x)\,d\alpha_1(x) - \int_a^b f_2(x)\,d\alpha_2(x)$$

$$+ i\left\{\int_a^b f_1(x)\,d\alpha_2(x) + \int_a^b f_2(x)\,d\alpha_1(x)\right\}$$

provided all the integrals on the R.H.S exist in the sence already dealt in before.

Note (1) If $\alpha(x) = x$, then $\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\,dx$, which is Riemann-integrals.

i.e. Stieltjes integral can be reduced to Riemann-integral taking particular value of $\alpha(x) = x$.

For this reason, the integral $\int_a^b f(x)\cdot d\alpha(x)$ is some-time, called Riemann-Stieltje's or R-S integral.

Note (2): If M_K = the least upper bound of $f(x)$ in I_K interval

m_K = the greatest lower bound of $f(x)$ in I_K interval

Then $\sum_{k=0}^{n-1} M_K[\alpha(x_{k+1}) - \alpha(x_K)]$ = [or Stieltji's upper sum] upper sum of $\alpha(x)$ in $[a,b]$

$= S_\Delta$, Δ is the partition

and $\sum_{k=0}^{n-1} m_K[\alpha(x_{k+1}) - \alpha(x_k)]$ = [or Stieltji's lower sum] lower sum of $\alpha(x)$ in $[a,b]$

$= s_\Delta$, Δ is the partition

V.V.I

Theorem If $f(x)$ and $\alpha(x)$ are real bounded functions in $a \le x \le b$ and $\alpha(x)$ is, in addition non-decreasing, then necessary and sufficient condition that $\int_a^b f(x)\,d\alpha(x)$ should exist is $\lim_{\delta \to 0} (S_\Delta - s_\Delta) = 0$, is independent of the manner of subdivision [R.U. 73, 74, 78]

[Find the necessary and sufficient-condition for R-S integrable]

: Condition is Necessary :- Given that $f(x), \alpha(x)$ are real bounded functions in $a \le x \le b$ and $\alpha(x)$ is non decreasing

Let $\int_a^b f(x)\,d\alpha(x)$ (i.e. Stieltje's integral) exists $= F$ (say)