Short Cut method to find PI

Let eqn be $f(D,D')z = \phi(x,y)$ ——— (1)

~~What~~ Which is homogeneous nth order PDE.

and let $\phi(x,y)$ be a function of $ax+by$.

Let $\phi(x,y) = \psi^n(ax+by)$ where $\psi^n(ax+by)$ is nth derivative of $\psi(ax+by)$ w.r to $ax+by$ then

$$PI = \frac{1}{f(D,D')}\psi^n(ax+by) = \frac{1}{f(a,b)}\psi(ax+by),$$

[Note: $\psi(ax+by)$ is obtained by integrating ϕ n times]

Exceptional Case: When $f(a,b)=0$

Then there must be a factor $(bD-aD')$ in $f(D,D')$

In this case

$$\frac{1}{(bD-aD')}\psi(ax+by) = \frac{x^n}{b^n\, n!}\psi(ax+by)$$

Ex (1) Solve: $(D^2-2DD'+D'^2)z = e^{x+2y}+x^3$

Sol^n: Here auxiliary equation is

$$m^2-2m+1=0$$
$$\Rightarrow (m-1)^2=0$$
$$\Rightarrow m=1,1$$

So complementary function is given by

$$CF = f_1(y+x) + x f_2(y+x).$$

Now, $$PI = \frac{1}{D^2-2DD'+D'^2}\left(e^{x+2y}+x^3\right)$$

$$= \frac{1}{D^2-2DD'+D'^2}e^{x+2y} + \frac{1}{D^2-2DD'+D'^2}x^3$$

$$= \frac{1}{(D-D')^2}e^{x+2y} + \frac{1}{(D-D')^2}x^3$$

$$= \frac{1}{(1-2)^2} e^{x+2y} + \frac{1}{D^2\left(1-\frac{D'}{D}\right)^2} x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2} x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left(1+\frac{2D'}{D}+\cdots\right)x^3$$

$$= e^{x+2y} + \left(\frac{1}{D^2}+\frac{2D'}{D^3}+\cdots\right)x^3$$

$$= e^{x+2y} + \frac{x^5}{20} + 0 + \cdots$$

$$= e^{x+2y} + \frac{x^5}{20}$$

Hence complete solution is

$$z = f_1(y+x) + x f_2(y+x) + e^{x+2y} + \frac{1}{20}x^5$$

Ans

Ex ② Solve: $(D^3 - 4D^2D' + 4DD'^2)z = 4\sin(2x+y)$

Solⁿ Here Auxiliary Equation is

$$m^3 - 4m^2 + 4m = 0$$

$$\Rightarrow m(m^2 - 4m + 4) = 0$$

$$\Rightarrow m(m-2)^2 = 0$$

$$\Rightarrow m = 0, 2, 2.$$

So complementary function is

$$CF = f_1(y) + f_2(y+2x) + x f_3(y+2x)$$

Now $PI = \frac{1}{D^3 - 4D^2D' + 4DD'^2} \cdot 4\sin(2x+y)$

$$= \frac{1}{D(D-2D')^2} 4\sin(2x+y)$$

$$= \frac{1}{(D-2D')^2} \cdot \frac{1}{D} 4\sin(2x+y)$$

$$= \frac{1}{(D-2D')^2} \cdot 4\left\{\frac{-\cos(2x+y)}{2}\right\} \quad \text{(integrating w.r.t. x)}$$

$$= \frac{1}{(D-2D')^2}\left\{-2\cos(2x+y)\right\}$$

$$= \frac{x^2}{1^2 \cdot \lfloor 2} \left\{-2\cos(2x+y)\right\}$$

$$= -x^2\cos(2x+y)$$

Hence complete solution is

$$z = f_1(y) + f_2(y+2x) + x f_3(y+2x) - x^2\cos(2x+y)$$

Ans

<u>For Practice</u>

1. Solve $(D^2 - 6DD' + 9D'^2)z = 6x + 2y$
2. Solve $(D^2 - 5DD' + 4D'^2)z = \sin(4x+y)$
3. Solve $4r - 4s + t = 16\log(x+2y)$
4. Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 12(x+y)$
5. Solve $2r - s - 3t = 5e^x/e^y$.