Then we have to prove that $\lim\limits_{\delta \to 0} (S_\Delta - s_\Delta) = 0$

where Δ = partition of $[a, b]$, δ = norm of the partition Δ

S_Δ = upper sum of $\alpha(x)$ for partition Δ

s_Δ = lower sum of $\alpha(x)$ for partition Δ

Let $\Delta = \{a = x_0, x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n = b\}$, be the partition of $[a, b]$

$I_k = [x_k, x_{k+1}]$, $k = 0, 1, 2, \dots (n-1)$ are the sub-interval for such partition

$\because f(x)$ is bounded in $[a, b]$

$\Rightarrow f(x)$ is bounded in each of I_k

Let M_k = lub of $f(x)$ in I_k

m_k = glb of $f(x)$ in I_k

then,
$$S_\Delta = \sum_{k=0}^{n-1} M_k [\alpha(x_{k+1}) - \alpha(x_k)]$$

and
$$s_\Delta = \sum_{k=0}^{n-1} m_k [\alpha(x_{k+1}) - \alpha(x_k)]$$

Now from definition of least upper bound,
$$f(x) \le M_k \quad , \forall x \in I_k$$

and $\exists$ at least one point say ξ_k in I_k such that
$$f(\xi_k) > M_k - \epsilon \quad , \text{for all } \epsilon > 0$$
$$\Rightarrow M_k < f(\xi_k) + \epsilon$$
$$\therefore \xi_k \in I_k \Rightarrow f(\xi_k) \le M_k < \{f(\xi_k) + \epsilon\}$$

$$\Rightarrow f(\xi_k)[\alpha(x_{k+1}) - \alpha(x_k)] \le M_k[\alpha(x_{k+1}) - \alpha(x_k)] < \{f(\xi_k) + \epsilon\} \cdot [\alpha(x_{k+1}) - \alpha(x_k)]$$

$$\Rightarrow \lim_{\delta \to 0} \sum_{k=0}^{n-1} f(\xi_k)[\alpha(x_{k+1}) - \alpha(x_k)] \le \lim_{\delta \to 0} \sum_{k=0}^{n-1} M_k[\alpha(x_{k+1}) - \alpha(x_k)]$$
$$< \lim_{\delta \to 0} \sum_{k=0}^{n-1} f(\xi_k)[\alpha(x_{k+1}) - \alpha(x_k)]$$
$$+ \lim_{\delta \to 0} \epsilon \sum_{k=0}^{n-1} [\alpha(x_{k+1}) - \alpha(x_k)]$$

$$\Rightarrow \quad F = \int_a^b f(x)\,d\alpha(x) \leq \lim_{\delta \to 0} s_\Delta < F + \epsilon[\alpha(b) - \alpha(a)] \text{ — } ①$$

$$\left\{ \because \int_a^b f(x)\,d\alpha(x) = \lim_{\delta \to 0} \sum_{k=0}^{n-1} f(\xi_k)[\alpha(x_{k+1}) - \alpha(x_k)] = F \right.$$

and $\alpha(x)$ is non-decreasing. so $\sum_{k=0}^{n-1} [\alpha(x_{k+1}) - \alpha(x_k)]$
$= [\alpha(b) - \alpha(a)]. \}$

Again m_k = $\underline{bd}$ of $f(x)$ in I_k

$\Rightarrow$ $f(x) \geq m_k \;\; \forall\, x \in I_k$

and $\exists$ a point ξ_k in I_k such that

$f(\xi_k) < m_k + \epsilon_1$ for all $\epsilon_1 > 0$

$\Rightarrow m_k > f(\xi_k) - \epsilon_1$

$\because \xi_k \in I_k \Rightarrow f(\xi_k) \geq m_k > f(\xi_k) - \epsilon_1$

Proceeding as above, we have

$$F \geq \lim_{\delta \to 0} s_\Delta > F - \epsilon_1[\alpha(b) - \alpha(a)]$$

$$\Rightarrow \quad -F \leq \lim_{\delta \to 0} (-s_\Delta) < \epsilon_1[\alpha(b) - \alpha(a)] - F \longrightarrow ②$$

Adding ① and ② we get

$$0 \leq \lim_{\delta \to 0} (S_\Delta - s_\Delta) < [\alpha(b) - \alpha(a)](\epsilon + \epsilon_1)$$

$$\therefore 0 \leq \lim_{\delta \to 0} (S_\Delta - s_\Delta) < \epsilon'$$

choosing $(\epsilon + \epsilon_1)[\alpha(b) - \alpha(a)] = \epsilon' > 0$ as $\alpha(x)$ is non-decreasing

$\Rightarrow \lim_{\delta \to 0} (S_\Delta - s_\Delta)$ exists.

And $0 \leq \lim_{\delta \to 0} (S_\Delta - s_\Delta) < \epsilon' \;\; \forall\, \epsilon' > 0$

$\Rightarrow \lim_{\delta \to 0} (S_\Delta - s_\Delta) = 0$, which is the necessary condition for $\int_a^b f(x)\,d\alpha(x)$ exists

<u>Sufficient condition</u>:

Let $\lim_{\delta \to 0} (S_\Delta - s_\Delta) = 0$, where the symbol have their usual meaning

Then we have prove that $\int_a^b f(x)\, d\alpha(x)$ exist, where $f(x)$ and $\alpha(x)$ are given real and bounded & $\alpha(x)$ is non-decreasing

If $\int_a^b f(x)\, d\alpha(x)$ exist only when

$$\lim_{\delta \to 0} \sum_{k=0}^{n-1} f(\xi_k)\left[\alpha(x_{k+1}) - \alpha(x_k)\right] \text{ will exist}$$

i.e. $\lim_{\delta \to 0} \sigma_\Delta$ exists where $\sigma_\Delta = \sum_{k=0}^{n-1} f(\xi_k)\left[\alpha(x_{k+1}) - \alpha(x_k)\right]$

Now, $\sigma_\Delta = \sum_{k=0}^{n-1} f(\xi_k)\left[\alpha(x_{k+1}) - \alpha(x_k)\right] \leq \sum_{k=0}^{n-1} M_k\left[\alpha(x_{k+1}) - \alpha(x_k)\right]$

but $\geq \sum_{k=0}^{n-1} m_k\left[\alpha(x_{k+1}) - \alpha(x_k)\right]$, $\{\because m_k \leq f(\xi_k) \leq M_k\}$

$$\Rightarrow \lim_{\delta \to 0} \sum_{k=0}^{n-1} m_k\left[\alpha(x_{k+1}) - \alpha(x_k)\right] \leq \lim_{\delta \to 0} \sigma_\Delta \leq \lim_{\delta \to 0} \sum_{k=0}^{n-1} M_k\left[\alpha(x_{k+1}) - \alpha(x_k)\right]$$

$$\Rightarrow \lim_{\delta \to 0} s_\Delta \leq \lim_{\delta \to 0} \sigma_\Delta \leq \lim_{\delta \to 0} S_\Delta$$

Since the given condition is $\lim_{\delta \to 0} (S_\Delta - s_\Delta) = 0$

$$\Rightarrow \lim_{\delta \to 0} S_\Delta = \lim_{\delta \to 0} s_\Delta$$

Hence, in order to prove, $\lim_{\delta \to 0} \sigma_\Delta$ exists, it is sufficient to prove that $\lim_{\delta \to 0} s_\Delta$ exists, for which it sufficies to prove $|s_{\Delta_1} - s_{\Delta_2}| < \epsilon$

where Δ_1, Δ_2 are arbitrary partitions of $[a, b]$

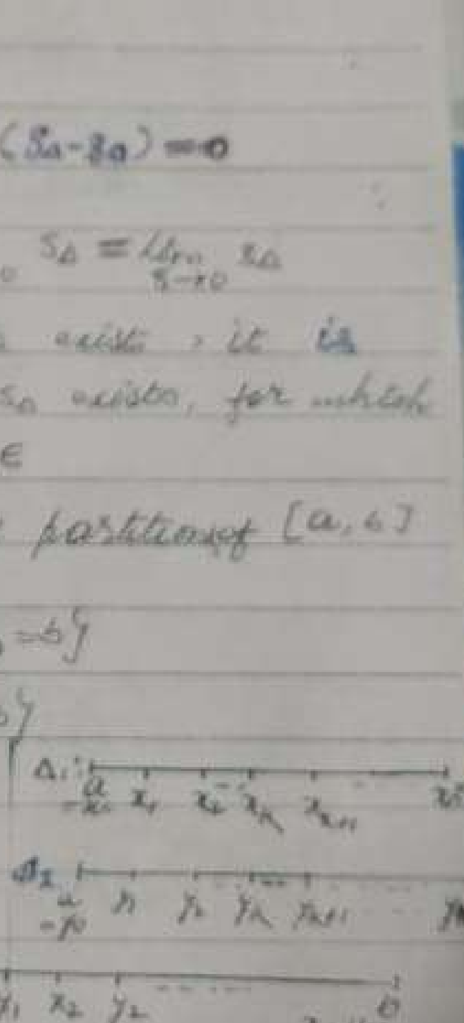
Let $\Delta_1 = \{a = x_0, x_1, x_2, \ldots, x_k, x_{k+1}, \ldots x_n = b\}$

$\Delta_2 = \{a = y_0, y_1, \ldots y_k, y_{k+1}, \ldots y_m = b\}$

be the partitions of $[a, b]$

we define another partition

$\Delta_3 = \Delta_1 \cup \Delta_2$ of $[a, b]$



Note: If $P = P_1 \vee P_2$ refinement partition, then $S_P \leq S_{P_1}$, $s_P \geq s_{P_1}$

clearly Δ_3 is refinement partition of Δ_1 and Δ_2, then we have

$$\left.\begin{aligned} s_{\Delta_1} \leq S_{\Delta_3} \leq S_{\Delta_1} \\ s_{\Delta_2} \leq S_{\Delta_3} \leq S_{\Delta_2} \end{aligned}\right\} \text{——} (A)$$

Given, $\lim_{\delta \to 0} (S_\Delta - s_\Delta) = 0$, where Δ is arbitrary

$\Rightarrow$ for any Δ, $|S_\Delta - s_\Delta| < \frac{\epsilon}{2}$, for $\delta > \delta_0(\epsilon)$

$\therefore$ for Δ_1, $|S_{\Delta_1} - s_{\Delta_1}| < \frac{\epsilon}{2}$; for $\delta_1 > \delta_0(\epsilon)$ —— (i)

for Δ_2, $|S_{\Delta_2} - s_{\Delta_2}| < \frac{\epsilon}{2}$; for $\delta_2 > \delta_0(\epsilon)$ —— (ii)

$\therefore S_{\Delta_1} < s_{\Delta_1} + \frac{\epsilon}{2} \leq S_{\Delta_3} + \frac{\epsilon}{2}$ {using (i) and (A)}

& $S_{\Delta_2} < s_{\Delta_2} + \frac{\epsilon}{2} \leq S_{\Delta_3} + \frac{\epsilon}{2}$ {using (ii) and (A)}

$$\begin{aligned} \therefore |S_{\Delta_1} - S_{\Delta_2}| &= |S_{\Delta_1} - S_{\Delta_3} + S_{\Delta_3} - S_{\Delta_2}| \\ &\leq |S_{\Delta_1} - S_{\Delta_3}| + |S_{\Delta_2} - S_{\Delta_3}| \\ &< |S_{\Delta_3} + \tfrac{\epsilon}{2} - S_{\Delta_3}| + |S_{\Delta_3} + \tfrac{\epsilon}{2} - S_{\Delta_3}| \quad \text{(by above eqn)} \\ &= \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus, $|S_{\Delta_1} - S_{\Delta_2}| < \epsilon$, for $\delta_1, \delta_2 > \delta_0(\epsilon)$

$\therefore \lim_{\delta \to 0} S_\Delta$ exists

So, $\lim_{\delta \to 0} \sigma_\Delta$ exists i.e. $\int_a^b f(x)\, d\alpha(x)$ exists

This prove the sufficient condition

~~Exact~~

—x— Function of Bounded Variation (b.v.) —x—

Let $\alpha(x)$, be a function defined over $I = [a, b]$ & let us consider a partition

$$\Delta = \{a = x_0, x_1, x_2 \ldots x_k, x_{k+1} \ldots x_n = b\} \text{ of } [a, b]$$