Some examples (solved)

(1) Solve $r + (a+b)s + abt = xy$

Sol^n Given differential equation can be written as

$$\left(D^2 + (a+b)DD' + abD'^2\right)z = xy$$

So auxilary eqn is

$$m^2 + (a+b)m + ab = 0$$

$$\Rightarrow m^2 + ma + mb + ab = 0$$

$$\Rightarrow m(m+a) + b(m+a) = 0$$

$$\Rightarrow (m+a)(m+b) = 0$$

$$\Rightarrow m = -a, -b$$

So complementary function is given by

$$CF = f_1(y - ax) + f_2(y - bx)$$

Now $PI = \dfrac{1}{D^2 + (a+b)DD' + abD'^2} \cdot xy$

$$= \frac{1}{D^2} \cdot \frac{1}{\left\{1 + (a+b)\frac{D'}{D} + ab\frac{D'^2}{D^2}\right\}} xy$$

$$= \frac{1}{D^2}\left\{1 + (a+b)\frac{D'}{D} + ab\frac{D'^2}{D^2}\right\}^{-1} xy$$

$$= \frac{1}{D^2}\left\{1 - (a+b)\frac{D'}{D} - ab\frac{D'^2}{D^2} + \cdots\right\} xy$$

$$= \left\{ \frac{1}{D^2} - (a+b)\frac{D'}{D^3} + \cdots \right\} xy$$

$$= \frac{1}{D^2} xy - (a+b)\frac{1}{D^3} \cdot x$$

$$= \frac{x^3 y}{6} - (a+b)\frac{x^4}{24}$$

Hence complete solution is

$$Z = f_1(y-ax) + f_2(y-bx) + \frac{x^3y}{6} - (a+b)\frac{x^4}{24}$$

Ans

General Method of finding PI

In this method we factorise the $f(D, D')$ in $\frac{1}{f(D D')} \phi(x,y)$ in linear factors & then we use following method

$$\frac{1}{D - mD'} \phi(x,y) = \int \phi(x, a-mx)\, dx$$

After the integration, replace a with $y+mx$.

For a clear idea let us solve some examples

Ex Solve $(D^2 - 2DD' - 15D'^2)z = 12xy$.

Solⁿ Auxiliary equation is

$$m^2 - 2m - 15 = 0$$

$$\Rightarrow m^2 - 5m + 3m - 15 = 0$$

$$\Rightarrow m(m-5) + 3(m-5) = 0$$

$$\Rightarrow (m-5)(m+3) = 0$$

$$\Rightarrow m = 5, -3$$

So $CF = f_1(y+5x) + f_2(y-3x)$

Now $PI = \frac{1}{D^2 - 2DD' - 15D'^2} 12xy$

$$= \frac{1}{(D-5D')(D+3D')} 12xy$$

$$= \frac{1}{D-5D'} \int 12x(a+3x)\,dx$$

$$= \frac{1}{D-5D'} \int (12ax + 36x^2)\,dx$$

$$= \frac{1}{D-5D'} (6ax^2 + 12x^3)$$

Putting $a = y - 3x$

$$= \frac{1}{D-5D'} \{6(y-3x)x^2 + 12x^3\}$$

$$= \frac{1}{D-5D'}\left(6x^2y - 18x^3 + 12x^3\right)$$

$$= \frac{1}{D-5D'}\left(6x^2y - 6x^3\right)$$

$$= \int \left\{6x^2(a-5x) - 6x^3\right\} dx$$

$$= \int \left(6ax^2 - 30x^3 - 6x^3\right) dx$$

$$= \int \left(6ax^2 - 36x^3\right) dx$$

$$= 2ax^3 - 9x^4$$

Putting $a = y + 5x$

$$= 2(y+5x)x^3 - 9x^4$$

$$= 2x^3y + 10x^4 - 9x^4$$

$$= x^4 + 2x^3y$$

Hence complete solution is

$$Z = f_1(y+5x) + f_2(y-3x) + x^4 + 2x^3y$$

Ans