

IMPORTANT THEOREMS

Theorem 1. If c_1 and c_2 are arbitrary constants, then $F\{c_1 f(x) + c_2 g(x)\} = c_1 F\{f(x)\} + c_2 F\{g(x)\}$.
(It is also called Linearity property)

Proof: $F\{c_1 f(x) + c_2 g(x)\} = \int_{-\infty}^{\infty} e^{-isx} \{c_1 f(x) + c_2 g(x)\} dx$

$$= \int_{-\infty}^{\infty} \{c_1 e^{-isx} f(x) + c_2 e^{-isx} g(x)\} dx$$

$$= \int_{-\infty}^{\infty} c_1 e^{-isx} f(x) dx + \int_{-\infty}^{\infty} c_2 e^{-isx} g(x) dx$$

$$= c_1 \int_{-\infty}^{\infty} e^{-isx} f(x) dx + c_2 \int_{-\infty}^{\infty} e^{-isx} g(x) dx$$

$$= c_1 F\{f(x)\} + c_2 F\{g(x)\} \quad \text{Proved.}$$

Theorem 2: If $f(s)$ is the Fourier transform of $F(x)$, then $\frac{1}{a} f\left(\frac{s}{a}\right)$ is the Fourier transform of $F(ax)$.
(change of scale property).

Proof: $\therefore F\{F(x)\} = \int_{-\infty}^{\infty} e^{-isx} F(x) dx = f(s).$

$$\therefore F\{F(ax)\} = \int_{-\infty}^{\infty} e^{-isx} F(ax) dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{ist}{a}} F(t) \frac{dt}{a}, \text{ where } ax = t$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} e^{-i\left(\frac{s}{a}\right)t} F(t) dt = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Proved.

Theorem 3: If $f_c(s)$ is the Fourier cosine transform of $F(x)$, then show that Fourier cosine transform of $F(\frac{x}{a})$ is $a f_c(as)$.

Proof:

$$\begin{aligned} F_c \left\{ F\left(\frac{x}{a}\right) \right\} &= \int_0^{\infty} F\left(\frac{x}{a}\right) \cos sx \, dx \\ \text{Put } \frac{x}{a} &= t \Rightarrow x = at \Rightarrow dx = a \, dt \\ &= \int_0^{\infty} F(t) \cos(sat) \cdot a \, dt \\ &= \int_0^{\infty} a F(t) \cos(sat) \, dt \\ &= a \int_0^{\infty} F(t) \cos(sat) \, dt = a f_c(as). \end{aligned}$$

Proved.

Theorem 4: If $f_s(s)$ is the Fourier sine transform of $F(x)$, then show that Fourier sine transform of $F(\frac{x}{a})$ is $a f_s(as)$.

Proof:

$$\begin{aligned} F_s \left\{ F\left(\frac{x}{a}\right) \right\} &= \int_0^{\infty} F\left(\frac{x}{a}\right) \sin sx \, dx \\ \text{Put } \frac{x}{a} &= t \Rightarrow x = at \Rightarrow dx = a \, dt \\ &= \int_0^{\infty} F(t) \sin(sat) \cdot a \, dt \\ &= a \int_0^{\infty} F(t) \sin(sat) \, dt \\ &= a f_s(as). \end{aligned}$$

Proved.

Theorem 5: If $f(s)$ is the Fourier transform of $F(x)$, then prove that $e^{ias} f(s)$ is the Fourier transform of $F(x-a)$. (Shifting property)

Proof: $F\{F(x-a)\} = \int_{-\infty}^{\infty} e^{-isx} F(x-a) dx$

Put $x-a = t \Rightarrow dx = dt$

$$= \int_{-\infty}^{\infty} e^{-is(a+t)} F(t) dt$$

$$= \int_{-\infty}^{\infty} e^{-ias} \cdot e^{-ist} F(t) dt$$

$$= e^{-ias} \int_{-\infty}^{\infty} e^{-ist} F(t) dt = e^{-ias} f(s)$$

Proved.

Theorem 6: If $F(x)$ has the Fourier transform $f(s)$, then $F(x) \cos ax$ has the Fourier transform $\frac{1}{2} f(s-a) + \frac{1}{2} f(s+a)$. (Modulation theorem)

Proof: Since we know that $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$$F\{F(x) \cos(ax)\} = \int_{-\infty}^{\infty} e^{-isx} F(x) \cos(ax) dx$$

$$= \int_{-\infty}^{\infty} e^{-isx} F(x) \frac{e^{iax} + e^{-iax}}{2} dx$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-i(s-a)x} F(x) dx + \int_{-\infty}^{\infty} e^{-i(s+a)x} F(x) dx \right]$$

$$= \frac{1}{2} [f(s-a) + f(s+a)]$$

Proved.