We define $V([a,b],\alpha,\pi) = \sum_{k=0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)|$

and $V([a,b],\alpha) = \sup V([a,b],\alpha,\pi)$

$= \sup \sum_{k=0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)|$

Supremum being taken over all partition of $[a,b]$

We say $V([a,b],\alpha)$ is the total variation of $\alpha(x)$ on $[a,b]$, then the function $\alpha(x)$ is said to be of bounded variation if $V([a,b],\alpha) < M$ (finite)

$\therefore$ A function $\alpha(x)$ is of bounded Variation on $[a,b]$ $\iff$ Total variation of $\alpha(x)$ on $[a,b]$ is bounded

Note $V([a,b],\alpha)$ is also sometime expressed by the ~~abbreviated~~ symbol $V_\alpha(b)$ or $V[\alpha]_a^b$

Thus $V_\alpha(t) \Rightarrow V_\alpha(t) = V([a,t]\alpha)$

—— x ——

Theorem 2(A) Every real functions of bounded variation on $[a,b]$ is bounded but not conversely

Proof:- Let $\alpha(x)$ be a real function of bounded variation on $[a,b]$

this $\Rightarrow V_\alpha(b)$ is bounded on $[a,b]$

$\Rightarrow \sup \sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)|$ is bounded

$\because$ Supremum being taken over all partition of $[a,b]$

$\Rightarrow \sup \sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)| \leq M$, M being a +ve number.

$\Rightarrow \sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)| \leq \sup \sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)| \leq M$

$\Rightarrow |\alpha(x_{k+1}) - \alpha(x_k)| < \sum_{k=0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)| \leq M$

Note: $\sin(n\pi - \theta) = (-1)^{n+1}\sin\theta$

$$\Rightarrow |\alpha(x_{k+1})| - |\alpha(x_k)| \le |\alpha(x_{k+1}) - \alpha(x_k)| \le M$$

$\{\because |A| - |B| \le |A - B|\}$

$$\Rightarrow |\alpha(x_{k+1})| - |\alpha(x_k)| \le M$$

$\Rightarrow |\alpha(x_{k+1})|, |\alpha(x_k)|$ both are finite.

$\because x_{k+1}, x_k$ being any two points in $[a, b]$

We can say $|\alpha(x)|$ is finite, $\forall\, x \in [a, b]$

$\Rightarrow \alpha(x)$ is bounded in $[a, b]$.

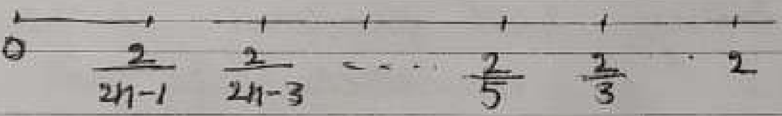
But the converse of above theorem is not true can be proved by the following example:

Let us consider $\alpha(x) = x\sin\frac{\pi}{x}$, $0 < x \le 2$

$= 0$, $x = 0$

clearly $\alpha(x)$ is bounded in $[0, 2]$

then we shall prove $\alpha(x)$ is not of bounded variation in $[a, b]$



Let us consider the partition $\Delta = \left\{0, \frac{2}{2n-1}, \frac{2}{2n-3}, \cdots, \frac{2}{5}, \frac{2}{3}, 2\right\}$ of $[0, 2]$

Now,
$$\sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)| = \left|\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)\pi - 0\right|$$

$$+ \left|\frac{2}{2n-3}\sin\frac{2n-3}{2}\pi - \frac{2}{2n-1}\sin\frac{2n-1}{2}\right|$$

$$+ \cdots + \left|\frac{2}{5}\sin\frac{5\pi}{2} - \frac{2}{7}\sin\frac{7\pi}{2}\right| + \left|\frac{2}{3}\sin\frac{3\pi}{2} - \frac{2}{5}\sin\frac{5}{2}\right|$$

$$+ \left|\frac{2}{1}\sin\frac{\pi}{2} - \frac{2}{3}\sin\frac{3}{2}\pi\right|$$

$$\le \frac{2}{2n-1} + \left(\frac{2}{2n-3} + \frac{2}{2n-1}\right) + \cdots + \left(\frac{2}{5} + \frac{2}{7}\right) + \left(\frac{2}{3} + \frac{2}{5}\right) + \left(2 + \frac{2}{3}\right)$$

$$= \frac{4}{2n-1} + \frac{4}{3n-1} + \cdots \frac{4}{3} + \frac{1}{2}$$

$\therefore \lim_{n\to\infty} \sum_{k=0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)| = \frac{4}{2n-1} + \frac{4}{2n-3} + \cdots$ (6)

$= \sum_{n=1}^{\infty} u_n$ (say)

~~Taking $n \to \infty$, then $\{4/3 \ldots\}$ ... to is divergent~~ series

Taking $v_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{u_n}{v_n} = 2$ (non-zero finite)

But $\sum v_n$ is auxiliary series, and it is divergent as $p=1$, hence by comparison test the series $\sum u_n$ is divergent.

i.e. $\lim_{n\to\infty} \sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)| >$ a divergent series

$\Rightarrow \sup \sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)| >$ a divergent series when $n \to \infty$

$\therefore V_\alpha(b) >$ a divergent series

$\Rightarrow V_\alpha(b)$ is not bounded.

So, the above example prove that a function is bounded may not be of bounded variation.

Theorem 2(B): If $\alpha(x)$ be a function of bounded variation on $[a,b]$, then $|\alpha(b) - \alpha(a)| \leq V([a,b],\alpha) = V_\alpha(b)$

Proof:

Let $\Delta = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a,b]$.

Clearly,
$$\sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)| \leq \sup \sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)|$$
$$= V_\alpha(b)$$

$$\Rightarrow \sum_{0}^{n-1} \{\alpha(x_{k+1}) - \alpha(x_k)\} \leq V_\alpha(b) \quad \text{(i)}$$

and
$$\sum_{0}^{n-1} -\{\alpha(x_{k+1}) - \alpha(x_k)\} \leq V_\alpha(b) \quad \text{(ii)}$$

$$\Rightarrow \{\alpha(b) - \alpha(a)\} \leq V_\alpha(b)$$

$\sum_{0}^{n-1} \{\alpha(x_{k+1}) - \alpha(x_k)\} = \alpha(b) - \alpha(a)$

and
$$\{\alpha(a) - \alpha(b)\} \leq V_\alpha(b)$$

$$\Rightarrow |\alpha(b) - \alpha(a)| \leq V_\alpha(b) = V([a,b],\alpha).$$

—— x ——

Theorem 2(C): A monotonic and bounded function on $[a,b]$ is of bounded variation on $[a,b]$, $V_\alpha(b) = |\alpha(b) - \alpha(a)|$.

Proof case I:-

Let $\alpha(x)$ be a function which is monotonic increasing and bounded on $[a,b]$

Let us consider a partition $\Delta = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a,b]$

Then, $\alpha(x)$ will be of bounded variation if $V_\alpha(b)$ exists

Now, $V_\alpha(b) = \sup \sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)|$

Supremum being taken over all partition of $[a,b]$

$$= \sup \sum_{0}^{n-1} \{\alpha(x_{k+1}) - \alpha(x_k)\}$$

[$\because$ since $\alpha(x)$ being increasing function]

$$= \sup\{\alpha(b) - \alpha(a)\}$$

$$= \alpha(b) - \alpha(a) = \text{a finite number}$$

this $\Rightarrow$ $V_\alpha(b)$ exists

$\Rightarrow$ $\alpha(x)$ is of bounded ~~and monotonic decreasing on [a,b]~~ variation in $[a,b]$.

case II:-

Let $\alpha(x)$ be bounded and monotonic decreasing on $[a,b]$, then

$$V_\alpha(b) = \sup \sum_{0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)|$$

$$= \sup \sum_{0}^{n-1} |-\{\alpha(x_k) - \alpha(x_{k+1})\}|$$

$$= \sup \sum_{0}^{n-1} |\alpha(x_k) - \alpha(x_{k+1})|$$

$$\therefore V_\alpha(b) = \sup \sum_{0}^{n-1} \{\alpha(x_k) - \alpha(x_{k+1})\},$$

Since $\alpha(x)$ is monotonic decreasing funct

$\Rightarrow V_\alpha(b) = \sup\{\alpha(a)-\alpha(b)\}$
$= \alpha(a)-\alpha(b)$, = a finite number

$\Rightarrow V_\alpha(b)$ exists on $[a,b]$

$\Rightarrow \alpha(x)$ is of bounded variation

Case III :-

Let $\alpha(x)$ is a constant function

then. $\alpha(x_{k+1}) = \alpha(x_k)$. for all k

$\Rightarrow \alpha(b) = \alpha(a)$

$\therefore V_\alpha(b) = \sup \sum_0^{n-1} |\alpha(x_{k+1})-\alpha(x_k)|$
$= 0$

$\Rightarrow V_\alpha(b)$ = exists

Combining all three cases, $V_\alpha(b)$ exist on $[a,b]$ when $\alpha(x)$ is bounded and monotonic function

$\Rightarrow \alpha(x)$ is of bounded variation

and $V_\alpha(b) = \alpha(b)-\alpha(a)$

$V_\alpha(b) = \alpha(a)-\alpha(b)$

$\therefore V_\alpha(b) = |\alpha(b)-\alpha(a)|$

Note :- (i) when $b=a$, then $V_\alpha(a) = |\alpha(a)-\alpha(a)| = 0$

$\therefore V([a,a],\alpha) = V_\alpha(a) = 0$

Remember :-

(i) If $\alpha(x)$ be a function of bounded variation on $[a$, then $\alpha(x)$ is ~~constant~~ continuous (B.S.C.)

$\Leftrightarrow V_\alpha(x)$ is continuous

(ii) $V([a,y],\alpha) = V([a,x],\alpha) + V([x,y],\alpha)$

if $a \le x \le y$

Imp Theorem 3(B) :- If $\alpha(x)$, be a real function of b.v. in $[a,b]$, then $\exists$ two functions $\alpha_1(x)$ and $\alpha_2(x)$, which non-negative, non-decreasing and bounded in $[a,$ which vanish at $x=a$, have the same point