Fourier Transform (3)

Theorem (7) Derivative theorem: The Fourier transform of $F'(x)$, the derivative of $F(x)$, is $isf(s)$, where $f(s)$ is the Fourier transform of $F(x)$.

Proof: By definition, $F\{F'(x)\} = \int_{-\infty}^{\infty} e^{-isx} F'(x)\,dx$.

ie $F\{F'(x)\} = \left[e^{-isx} F(x)\right]_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} (-is) e^{-isx} F(x)\,dx$

$= 0 + is\int_{-\infty}^{\infty} F(x)\, e^{-isx}\,dx$ $(\because F(x)\to 0$ as $x\to\pm\infty)$

$= isf(s)$

$\therefore F\{F'(x)\} = isf(s)$ if $F(x)\to 0$ as $x\to\pm\infty$.

Theorem (8) To show that

$$F\left\{\frac{d^n F(x)}{dx^n}\right\} = (is)^n f(s), \text{ where } F\{F(x)\} = f(s),$$

if the first $(n-1)$ derivatives of $F(x)$ vanish identically as $x\to\pm\infty$.

Proof: Let first $(n-1)$ derivatives of $F(x)$ vanish as $x\to\pm\infty$

$$F\left\{\frac{d^n F}{dx^n}\right\} = \int_{-\infty}^{\infty} e^{-isx} \frac{d^n F}{dx^n}\,dx \text{, Integrating by parts}$$

$$= \left[e^{-isx}\frac{d^{n-1}F}{dx^{n-1}}\right]_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} (-is) e^{-isx} \frac{d^{n-1}F}{dx^{n-1}}\,dx$$

$$= 0 + is\int_{-\infty}^{\infty} e^{-isx}\frac{d^{n-1}F}{dx^{n-1}}\,dx = (is)^1 F\left\{\frac{d^{n-1}F}{dx^{n-1}}\right\}$$

Dr. S.D. [illegible]
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Again, $F\left\{\frac{d^nF}{dx^n}\right\} = is\int_{-\infty}^{\infty} e^{-isx}\frac{d^{n-1}F}{dx^{n-1}}\,dx$,
Integrating by parts, we have

$$F\left\{\frac{d^nF}{dx^n}\right\} = (is)\left[\left\{e^{-isx}\frac{d^{n-2}F}{dx^{n-2}}\right\}_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty}(-is)e^{-isx}\frac{d^{n-2}F}{dx^{n-2}}\,dx\right]$$

$$= is.0 + (is)^2\int_{-\infty}^{\infty} e^{-isx}\frac{d^{n-2}F}{dx^{n-2}}\,dx$$

$$= (is)^2 F\left\{\frac{d^{n-2}F}{dx^{n-2}}\right\}.$$

Similarly, we can show that

$$F\left\{\frac{d^nF}{dx^n}\right\} = (is)^n F\{F(x)\} = (is)^n f(s),$$

where $F(x) = F$. **Proved.**

Definition: Convolution or falting: The convolution of two functions $F(x)$ and $G(x)$, where $-\infty < x < \infty$, is denoted and defined as

$$F * G = \int_{-\infty}^{\infty} F(u)\,G(x-u)\,du \quad \text{or,} \quad F*G = \int_{-\infty}^{\infty} G(u)\,F(x-u)\,du$$

Theorem (9) Convolution theorem for Fourier transform.

The Fourier transform of the convolution of $F(x)$ and $G(x)$ is the product of the Fourier transforms of $F(x)$ and $G(x)$ ie

$$F\{F * G\} = F\{F(x)\}.\,F\{G(x)\}$$

Proof: $LHS = F\{F*G\} = F\left\{\int_{-\infty}^{\infty} F(u)\,G(x-u)\,du\right\}$

$$= \int_{-\infty}^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} F(u)\,G(x-u)\,du\right\}, \text{ by definition of F.T.}$$

$$= \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} e^{isx}\, G(x-u)\, dx\right\} du, \text{ (by changing the order of integration)}$$

$$= \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} e^{is(u+v)}\, G(v)\, dv\right\} du, \text{ Putting } x-u=v,\ dx=dv$$

$$= \int_{-\infty}^{\infty} e^{isu} F(u)\left\{\int_{-\infty}^{\infty} e^{isv}\, G(v)\, dv\right\} du$$

$$= \int_{-\infty}^{\infty} e^{isu} F(u)\, F\{G(v)\}\, du, \text{ by definition}$$

$$= \int_{-\infty}^{\infty} e^{isu} F(u)\, F\{G(x)\}\, du$$

$$= F\{G(x)\} \int_{-\infty}^{\infty} e^{isu} F(u)\, du$$

$$= F\{G(x)\} \cdot F\{F(u)\}, \text{ by definition}$$

$$= F\{G(x)\} \cdot F\{F(x)\} = F\{F(x)\} \cdot F\{G(x)\}.$$

Proved.

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