continuity and discontinuity as $\alpha(x)$ and are such that $\alpha(x) - \alpha(a) = \alpha_1(x) - \alpha_2(x)$

$V_\alpha(x) = \alpha_1(x) + \alpha_2(x), \quad a \le x \le b$

Proof:- We define α_1 and α_2 as follows

$$\alpha_1(x) = \frac{1}{2}[V_\alpha(x) + \alpha(x) - \alpha(a)] \text{ —— } \langle i \rangle$$

$$\alpha_2(x) = \frac{1}{2}[V_\alpha(x) - \alpha(x) + \alpha(a)] \text{ —— } \langle ii \rangle$$

Also, we know that, $|\alpha(x) - \alpha(a)| \le V_\alpha(x)$

$$\Rightarrow \begin{cases} \alpha(x) - \alpha(a) \le V_\alpha(x) \\ \text{and } \alpha(a) - \alpha(x) \le V_\alpha(x) \end{cases}$$

Hence, $\alpha_1(x)$ and $\alpha_2(x)$ are non-negative by condition $\langle i \rangle$ and $\langle ii \rangle$.

Since $\alpha(x)$ is of bounded variation on $[a, b]$

$\Rightarrow \alpha(x)$ is bounded and $V_\alpha(x)$ exists

$\Rightarrow \alpha(x), V_\alpha(x)$ both bounded

$\therefore \alpha_1(x)$ is bounded by condition $\langle i \rangle$ and $\alpha_2(x)$ is bounded by condition $\langle ii \rangle$

When $x = a$

then $\alpha_1(a) = \frac{1}{2}[V_\alpha(a) + \alpha(a) - \alpha(a)]$

$= 0 \quad \{\because V_\alpha(a) = 0\}$

Similarly, $\alpha_2(a) = 0$

i.e. $\alpha_1(x)$ and $\alpha_2(x)$ vanish at $x = a$

Moreover, $\alpha(x)$ is discontinuous or continuous

$\Rightarrow \alpha_1(x)$ and $\alpha_2(x)$ both are discontinuous or c

Hence, $\alpha_1(x)$ and $\alpha_2(x)$ have the same point of continuity and discontinuity as $\alpha(x)$

Next.

$$\alpha_1(x) - \alpha_2(x) = \frac{1}{2}[V_\alpha(x) + \alpha(x) - \alpha(a) - V_\alpha(x) + \alpha(x) - \alpha(a)]$$

$$\Rightarrow \alpha_1(x) - \alpha_2(x) = \alpha(x) - \alpha(a)$$

and $\alpha_1(x) + \alpha_2(x) = \frac{1}{\cancel{2}} \times \cancel{2}\, V_\alpha(x) = V_\alpha(x)$

Lastly we are left to prove $\alpha_1(x)$ and $\alpha_2(x)$ are non-decreasing

Taking $a \le x < y \le b$

$\therefore \alpha_1(y) - \alpha_1(x) = \frac{1}{2}[V_\alpha(y) + \alpha(y) - \alpha(a) - V_\alpha(x) - \alpha(x) + \alpha(a)]$

$= \frac{1}{2}[V_\alpha(x) + V([x,y],\alpha) + \alpha(y) - V_\alpha(x) - \alpha(x)]$

$\{\because V([a,y],\alpha) = V([a,b],\alpha) + V([x,y],\alpha),$ if $a \le x < y \le b\}$

$\Rightarrow \alpha_1(y) - \alpha_1(x) = \frac{1}{2}[V([x,y],\alpha) + \alpha(y) - \alpha(x)]$

Similarly

$\alpha_2(y) - \alpha_2(x) = \frac{1}{2}[V([x,y],\alpha) - \alpha(y) + \alpha(x)]$

Since, $|\alpha(y) - \alpha(x)| \le V([x,y],\alpha)$

$\Rightarrow \alpha(y) - \alpha(x) \le V([x,y],\alpha)$

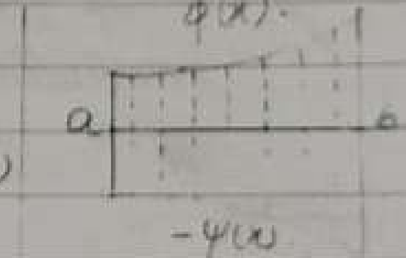
and $\alpha(x) - \alpha(y) \le V([x,y],\alpha)$

$\therefore$ from above we see

$$\left.\begin{array}{l}\alpha_1(y) - \alpha_1(x) \ge 0 \\ \text{and } \alpha_2(y) - \alpha_2(x) \ge 0\end{array}\right\} \text{ when } y \ge x$$

$\therefore \alpha_1(x)$ and $\alpha_2(x)$ are non-decreasing in $[a,b]$

Remark :-



$\alpha(x) - \alpha(a) = \alpha_1(x) - \alpha_2(x)$

$\Rightarrow \alpha(x) = [\alpha_1(x) + \alpha(a)] - \alpha_2(x)$

$\Rightarrow \alpha(x) = \varphi(x) - \psi(x)$

where $\varphi(x) = \alpha_1(x) + \alpha(a)$ & $\psi(x) = \alpha_2(x)$

Also, $\alpha_1(x), \alpha_2(x)$ are non-negative ; non-decreasing

$\Rightarrow \varphi(x), \psi(x)$ are so

Hence, A funct. $\alpha(x)$ of bounded variation on $[a,b]$ can always be represented into the form $\alpha(x) = \varphi(x) - \psi(x)$ where φ, ψ are non-decreasing and non-negative

Imp. Theorem 3(c)

If $\alpha_1(x)$ and $\alpha_2(x)$ are monotonic and bounded in $[a,b]$ and if $\alpha(x) = \alpha_1(x)$ and $\alpha_2(x)$, then

$$V_\alpha(x) \le |\alpha_1(x) - \alpha_1(a)| + |\alpha_2(x) - \alpha_2(a)|$$

Proof :- Since $\alpha_1(x)$ and $\alpha_2(x)$ are monotonic and bounded in $[a,b]$

$$\Rightarrow V_{\alpha_1}(x) = |\alpha_1(x) - \alpha_1(a)| \quad \text{[by theorem 3(b)]}$$

and $V_{\alpha_2}(x) = |\alpha_2(x) - \alpha_2(a)|$

$$\therefore V_{\alpha_1}(x) + V_{\alpha_2}(x) = |\alpha_1(x) - \alpha_1(a)| + |\alpha_2(x) - \alpha_2(a)|$$

$$\because \alpha(x) = \alpha_1(x) \Rightarrow V_\alpha(x) = V_{\alpha_1}(x)$$

and $\alpha(x) = \alpha_2(x) \Rightarrow V_\alpha(x) = V_{\alpha_2}(x)$

$$\therefore V_\alpha(x) + V_\alpha(x) = |\alpha_1(x) - \alpha_1(a)| + |\alpha_2(x) - \alpha_2(a)|$$

$$\Rightarrow 2V_\alpha(x) = |\alpha_1(x) - \alpha_1(a)| + |\alpha_2(x) - \alpha_2(a)|$$

$$\Rightarrow V_\alpha(x) \le 2V_\alpha(x) = |\alpha_1(x) - \alpha_1(a)| + |\alpha_2(x) - \alpha_2(a)|$$

V.I. V.V.I.

Existence Theorem 4(a) :-

If $f(x)$ is continuous and $\alpha(x)$ is of bounded variation in $[a,b]$, then the Stieltjes integral of $f(x)$ w.r.t. $\alpha(x)$ from a to b exists.

Proof :-

Given $f(x)$ is continuous and $\alpha(x)$ is of bounded variation in $[a,b]$

To prove Stieltje's integral of $f(x)$ w.r.t. from a to b exists i.e. $\int_a^b f(x)\,d\alpha(x)$ exists

Let us assume that $f(x)$ and $\alpha(x)$ are real functions

As, $\alpha(x)$ is of bounded variation in $[a,b]$, we