(4)

can write $\alpha(x) = \phi(x) - \psi(x)$
where ϕ and ψ are bounded and non-decreasing functions in $[a, b]$

Hence, we may take $\alpha(x)$ as bounded & non-decreasing

Since $f(x)$ is continuous in $[a, b]$, so $f(x)$ is bounded in $[a, b]$

Thus we have $f(x), \alpha(x)$ are real and bounded and in addition $\alpha(x)$ is non-decreasing in $[a, b]$

Hence, in order to prove that $\int_a^b f(x) d\alpha(x)$ exists, it is sufficient to prove

$$\lim_{\delta \to 0} (S_\Delta - s_\Delta) = 0$$

where Δ = any partition of $[a, b]$

S_Δ = upper sum of $\alpha(x)$ in $[a, b]$

s_Δ = lower sum of $\alpha(x)$ in $[a, b]$

δ = Norm of partition Δ

Now,

$$S_\Delta - s_\Delta = \sum_{k=0}^{n-1} M_k [\alpha(x_{k+1}) - \alpha(x_k)] - \sum_{k=0}^{n-1} m_k [\alpha(x_{k+1}) - \alpha(x_k)]$$

where M_k = $\overline{bd}$ of $f(x)$ in δ_k & m_k = $\underline{bd}$ of $f(x)$ in δ_k, and $\delta_k = [x_k, x_{k+1}]$ sub interval of $[a, b]$.

$$\therefore S_\Delta - s_\Delta = \sum_{k=0}^{n-1} (M_k - m_k) [\alpha(x_{k+1}) - \alpha(x_k)]$$

$$\therefore |S_\Delta - s_\Delta| = \left| \sum_{k=1}^{n} (M_k - m_k) [\alpha(x_{k+1}) - \alpha(x_k)] \right|$$

$$\Rightarrow |S_\Delta - s_\Delta| \leq \sum_{k=1}^{n-1} |M_k - m_k| \cdot |\alpha(x_{k+1}) - \alpha(x_k)|$$

As $f(x)$ is continuous in $[a, b]$, it is uniformly continuous there $\Rightarrow \epsilon > 0$, $\exists$ a +ve integer δ_0, such that

$$|M_k - m_k| < \epsilon, \text{ for } \delta < \delta_0$$

∴ from above

$$|S_\Delta - s_\Delta| \le \sum_{k=0}^{n-1} \epsilon |\alpha(x_{k+1}) - \alpha(x_k)| \quad , \quad \delta < \delta_0$$

$$= \epsilon \sum_{k=0}^{n-1} \{\alpha(x_{k+1}) - \alpha(x_k)\}$$

{~~∵ mod is immaterial here, since α(x) is nondecreasing~~
~~∴ α(x_{k+1}) ≥ α(x_k)~~}

$$= \epsilon [\alpha(b) - \alpha(a)] \quad \{\because \alpha(x) \text{ is monotonic non-decreasing}\}$$

$$\Rightarrow |S_\Delta - s_\Delta| \le \epsilon_1 \quad , \text{ choosing } \epsilon = \frac{\epsilon_1}{\alpha(b) - \alpha(a)} \ge 0$$

∴ $\forall\ \epsilon_1 > 0$, $\exists$ a +ve integer δ_0 such that

$$|S_\Delta - s_\Delta| < \epsilon_1 \quad , \text{ for } \delta < \delta_0$$

$$\Rightarrow \lim_{\delta \to 0} (S_\Delta - s_\Delta) = 0$$

$\Rightarrow f(x)$ is Stieltjes integrable w.r.t. $\alpha(x)$ in $[a, b]$

i.e. $\int_a^b f(x)\, d\alpha(x)$ exists

VI

Integration by parts formulae in Stieltjes integral

Theorem 4(b):- If $f(x)$ is of bounded variation and $\alpha(x)$ is continuous in $[a, b]$, then the Stieltjes integral of $f(x)$ w.r.t $\alpha(x)$ from a to b exists and

$$\int_a^b f(x)\, d\alpha(x) = f(b)\cdot\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\, df$$

Proof:-

Let us consider a partition $\Delta = \{a = x_0, x_1, x_2, \ldots, x_k, x_{k+1}, \ldots\ldots, x_n = b\}$ of $[a, b]$

Taking points $\xi_0, \xi_1, \xi_2, \ldots, \xi_{n-1}$ in the sub-interval $I_0, I_1, I_2, \ldots, I_k, \ldots I_{n-1}$ respectively where $I_k = [x_k, \ldots$

Let us consider the sum

$$\sigma_\Delta = \sum_{k=0}^{n-1} f(\xi_k) [\alpha(x_{k+1}) - \alpha(x_k)]$$

$= f(\xi_0)[\alpha(x_1)-\alpha(x_0)] + f(\xi_1)[\alpha(x_2)-\alpha(x_1)]$

$\quad + f(\xi_2)[\alpha(x_3)-\alpha(x_2)] + \cdots + f(\xi_{n-1})[f(x_n)-\alpha(x_{n-1})]$

$= \alpha(x_1)[f(\xi_0)-f(\xi_1)] + \alpha(x_2)[f(\xi_1)-f(\xi_2)] + \cdots + \alpha(x_k)[f(\xi_{k-1}) - f(\xi_k)]$

$\quad + \alpha(x_{k+1})[f(\xi_k) - f(\xi_{k+1})] + \cdots + \alpha(x_{n-1})[f(\xi_{n-2}) - f(\xi_{n-1})]$

$\quad - \alpha(x_0) f(\xi_0) + \alpha(x_n) f(\xi_{n-1})$

$= -\alpha(x_1)[f(\xi_1)-f(\xi_0)] - \alpha(x_2)[f(\xi_2)-f(\xi_1)] + \cdots +$

$\quad - \alpha(x_{n-1})[f(\xi_{n-1}) - f(\xi_{n-2})] - \alpha(a) f(\xi_0) + \alpha(b) f(\xi_{n-1})$

$= \alpha(b) f(\xi_{n-1}) - \alpha(a) f(\xi_0) - \sum_{k=1}^{n-1} \alpha(x_k)[f(\xi_k) - f(\xi_{k-1})]$

$= \alpha(b)[f(\xi_{n-1}) - f(b)] - \alpha(a)[f(\xi_0) - f(a)]$

$\quad - \sum_{k=1}^{n-1} \alpha(x_k)[f(\xi_k) - f(\xi_{k-1})] + \alpha(b) f(b) - \alpha(a) f(a)$

Now, taking the number of sub-intervals in $[a,b]$ by the partitions Δ infinitely ie $\delta \to 0$

ie. Max of $\{x_1 - a, x_2 - x_1, \ldots, b - x_{n-1}\} \to 0$

$\Rightarrow$ $\xi_0 - a$, $\xi_1 - x_1$, $\xi_2 - x_2$, ---, $b - \xi_{n-1}$ all are tends to 0

$\xi_0 \to a$, $\xi_1 \to x_1$, $\xi_2 \to x_2$, ---, $\xi_{n-1} \to b$

$f(\xi_0) \to f(a)$, $f(\xi_1) \to f(x_1)$, ..., $f(\xi_{n-1}) \to f(b)$

$\therefore$ from above,

$$\lim_{\delta \to 0} \sigma_\Delta = f(b)\alpha(b) - f(a)\alpha(a) - \lim_{\delta \to 0} \sum_{1}^{n-1} \alpha(x_k)[f(\xi_k) - f(\xi_{k-1})]$$

As, $\alpha(x)$ is continuous and $f(x)$ is of b.v. in $[a,b]$

$\therefore \lim_{\delta \to 0} \sum_{1}^{n-1} \alpha(x_k)[f(\xi_k) - f(\xi_{k-1})]$ exists and equal to

$$\int_a^b \alpha(x)\, df(x)$$

$\Rightarrow \lim_{\delta \to 0} \sigma_\Delta$ also exists in $[a,b]$ and equal to

$$\int_a^b f(x)\, d\alpha(x)$$

$$\therefore \int_a^b f(x)\, d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\, df(x).$$

Remark:-

When $\int_a^b f(x)\, d\alpha(x)$ exist, then

$$\lim_{\delta \to 0} s_\Delta = \lim_{\delta \to 0} S_\Delta$$

and $\int_a^b f(x)\, d\alpha(x) = \lim_{\delta \to 0} s_\Delta = \lim_{\delta \to 0} S_\Delta$.

Properties of Stieltjes integral

Theorem 5(b):- If $f(x)$ is continuous and $\alpha(x)$ is of b.v. in $a \leq x \leq b$, then for any c in $a < c < b$

(i) $\int_a^b f(x)\, d\alpha(x) = \int_a^c f(x)\, d\alpha(x) + \int_c^b f(x)\, d\alpha(x)$.

(ii) $\left| \int_a^b f(x)\, d\alpha(x) \right| \leq \int_a^b |f(x)|\, dV_\alpha(x) \leq \max_{a \leq x \leq b} |f(x)| \cdot V_\alpha(b)$

Proof of (i):-

Given $f(x)$ is continuous and $\alpha(x)$ is of b.v. in $[a,b]$

$\Rightarrow$ $f(x)$ is continuous and $\alpha(x)$ is of b.v. in each of $[a,c]$, $[c,b]$ where $a < c < b$.

$\int_a^c f(x)\, d\alpha(x)$, $\int_c^b f(x)\, d\alpha(x)$, $\int_a^b f(x)\, d\alpha(x)$ are all exists.