<u>Theorem:</u> Let F be field of quotients of a UFD "R". If $f(x) \in R[x]$ is both primitive and irreducible as an element of $R[x]$, then it is irreducible as an element of $F[x]$. Conversely, if the primitive element $f(x)$ in $R[x]$ is irreducible as an element of $F[x]$, it is also irreducible as an element of $R[x]$.

<u>Proof:</u> (Necessary part) Let $f(x)$ be a primitive member of $R[x]$ and let $f(x)$ is irreducible in $R[x]$. We want show that $f(x)$ is irreducible in $F[x]$.

Let us assume to the contradiction that $f(x)$ is reducible in $F[x]$.

$\Rightarrow \exists$ $g(x)$ & $h(x) \in F[x]$ such that

$$f(x) = g(x).h(x). \quad \text{——— ①}$$

Since $g(x)$ & $h(x) \in F[x]$,

$\Rightarrow \exists$ $a, b \in R$ & $g_0(x)$, $h_0(x) \in R[x]$

such that $g(x) = \frac{g_0(x)}{a}$ & $h(x) = \frac{h_0(x)}{b}$.

Also $g_0(x) = \alpha g_1(x)$ & $h_0(x) = \beta h_1(x)$

where $\alpha = c(g_0(x))$ & $\beta = c(h_0(x))$

and $g_1(x)$ & $h_1(x)$ are primitive members of $R[x]$

Thus $f(x) = \frac{\alpha\beta}{ab}\, g_1(x).h_1(x)$ [By Putting in ①]

$\Rightarrow ab\, f(x) = \alpha\beta\, g_1(x).h_1(x)$

Since $g_1(x)$ & $h_1(x)$ are primitive members of $R[x]$

$\Rightarrow g_1(x).h_1(x)$ is also a primitive in $R[x]$

$\Rightarrow f(x)$ & $g_1(x)h_1(x)$ are associates in $R[x]$

$\Rightarrow \exists$ a unit u in $R[x]$ (Which is also a unit of R)

Such that $f(x) = u.g_1(x).h_1(x)$

$\Rightarrow f(x) = g_2(x).h_1(x)$ when $g_2(x) = u\, g_1(x)$

Since $\deg(g_2(x)) = \deg(g(x))$

& $\deg(h_1(x)) = \deg(h(x))$

$\Rightarrow \deg(g_2(x)) > 0$ & $\deg(h_1(x)) > 0$

$\Rightarrow$ Neither $g_2(x)$ nor $h_1(x)$ is unit in $R[x]$

$\Rightarrow f(x) = g_2(x).h_1(x)$ is proper factorisation of $f(x)$ in $R[x]$. Which is a contradiction.

Hence $f(x)$ must be irreducible in $F[x]$

Sufficient part: Let $f(x)$ is primitive in $R[x]$ and $f(x)$ is irreducible as an element of $F[x]$. We want to show that $f(x)$ is irreducible in $R[x]$.

Let $f(x) = g(x) \cdot h(x)$ when $g(x)$ & $h(x) \in R[x]$

To show that $f(x)$ is irreducible, we have to show that either $g(x)$ or $h(x)$ is unit in $R[x]$ ie either $g(x)$ & $h(x)$ is unit in R

Since $g(x)$ & $h(x) \in R[x]$, we can write $g(x)$ & $h(x) \in F[x]$

Since $f(x)$ is irreducible in $F[x]$

$\Rightarrow$ Either $g(x)$ or $h(x)$ must be of degree zero.

Let us suppose that $\deg(g(x)) = 0$

$\Rightarrow g(x)$ is constant polynomial so let

$$g(x) = k.$$

$$\Rightarrow f(x) = k \cdot h(x)$$

Since $f(x)$ is primitive in R

$\Rightarrow c(f(x))$ is unit in R.

$\Rightarrow k$ is unit in R, otherwise if k is not unit in R then $c(k \cdot h(x))$ cannot be unit in R so it will not be equal to $c(f(x))$

(Hence proved)