Theorem: Let F be field of quotients of a unique factorisation domain R. If $f_1(x)$ & $f_2(x)$ are two primitive members of $R[x]$ and are associates in $F[x]$, then they are associates in $R[x]$ also.

Proof: Since $f_1(x)$ & $f_2(x)$ are associates in $F[x]$

$\Rightarrow \exists$ a unit $k \in F[x]$ such that

$$f_1(x) = k\, f_2(x) \quad \text{——— ①}$$

We know that units of $F[x]$ are the non-zero elements of F. Since F is field of quotients of R and $k \neq 0$

$$\Rightarrow k = \frac{g}{h} \quad \text{where } g, h \in R, \ h \neq 0$$

Putting in ① we get

$$f_1(x) = \frac{g}{h} f_2(x)$$

$$\Rightarrow h f_1(x) = g f_2(x) \quad \text{——— ②}$$

Since $f_1(x)$ & $f_2(x)$ are primitive polynomials in $R[x]$ so from ② we conclude that $f_1(x)$ & $f_2(x)$ are associates in $R[x]$

Theorem: If R is UFD and $p(x)$ is primitive member of $R[x]$, then it can be factorised in a unique way as the product of irreducible elements in $R[x]$.

Hence show that the polynomial ring $R[x]$ over a UFD 'R' is itself a UFD.

Proof Let F be field of quotients of R. where R is UFD. Let $p(x)$ be primitive member of $R[x]$. We can regard $p(x)$ as a member of $F[x]$. Since F is a field, therefore $F[x]$ is UFD. Therefor $p(x) \in F[x]$ can be factored as

$$p(x) = p_1(x) \cdot p_2(x) \cdots p_K(x) \quad \text{——} (1)$$

where $p_1(x), p_2(x), \cdots p_K(x)$ are irreducible polynomials in $F[x]$

Now each polynomial $p_i(x)$ for $1 \leq i \leq K$ can be written as

$$p_i(x) = \frac{f_i(x)}{a_i} \quad \text{where } a_i \in R \ \forall i$$

Further $f_i(x)$ can be written as

$$f_i(x) = b_i q_i(x) \quad \text{where } b_i \in R$$

and $q_i(x)$ is primitive in $R[x]$

Thus $p_i(x) = \frac{b_i}{a_i} q_i(x)$ for $1 \leq i \leq k$

where $b_i, a_i \in R$ and $q_i(x)$ is primitive in $R[x]$.

Since $p_i(x)$ is irreducible in $F[x]$ so $q_i(x)$ is also irreducible in $F[x]$.

Now $q_i(x)$ is primitive in $R[x]$ and irreducible in $F[x]$, therefore $q_i(x)$ is irreducible in $R[x]$

Now $p(x) = p_1(x) \cdot p_2(x) \cdots p_k(x)$ (From ①)

$$p(x) = \frac{b_1 b_2 \cdots b_k}{a_1 a_2 \cdots a_k} q_1(x) q_2(x) \cdots q_k(x)$$

$$\Rightarrow a_1 a_2 \cdots a_k p(x) = b_1 b_2 \cdots b_k q_1(x) \cdot q_2(x) \cdots q_k(x) \quad \text{—— ②}$$

Since $q_1(x), q_2(x), \cdots q_k(x)$ are primitive in $R[x]$

$\Rightarrow$ Product $q_1(x) \cdot q_2(x) \cdots q_k(x)$ is also primitive in $R[x]$

Also $p(x)$ on LHS of ② is primitive, hence from eqn ② we conclude that $p(x)$ and $q_1(x) \cdot q_2(x) \cdots q_k(x)$ are associates in $R[x]$

$$\text{ie } p(x) = u\, q_1(x) \cdot q_2(x) \cdots q_k(x) \quad \text{—— ③}$$

where u is unit in $R[x]$

If we replace $u \cdot q_1(x)$ with $q_1(x)$ we get

$$p(x) = q_1(x) \cdot q_2(x) \cdots q_k(x) \quad \text{—— ④}$$

Thus polynomial in $R[x]$ can be represented as product of irreducible elements.

Now, we want to show that the factorisation of $p(x)$ as in eqn ④ is unique upto the order and associates of irreducible elements.

Let $p(x) = r_1(x)\, r_2(x) \cdots r_m(x)$. where $r_j(x)$ are irreducible in $R[x]$ $\forall$ $1 \leq j \leq m$.

Since $p(x)$ is primitive, therefore each $r_j(x)$ is primitive. Hence $r_j(x)$ is irreducible in $F[x]$. But $F[x]$ is UFD therefore $p(x) \in F[x]$ can be uniquely expressed as product of irreducible elements of $F[x]$. Hence $r_j(x)$ & $q_i(x)$ regarded as elements of $F[x]$ are equal (upto associates) in some order.

Since $r_j(x)$ & $q_i(x)$ are primitive members of $R[x]$ and associates in $F[x]$, therefore they are also associates in $R[x]$. (First part prove)

Now, we will prove that $R[x]$ is also UFD.

Let $f(x) \in R[x]$.

Then we can write
$$f(x) = c.g(x) \text{ where } c \in R \text{ \& } g(x)$$
is primitive in $R[x]$.

Now as proved above $g(x)$ can be uniquely expressed as product of irreducible elements in $R[x]$.

Let $c = h_1(x).h_2(x) \ldots h_s(x)$ where $h_1(x), h_2(x) \ldots \in R[x]$

Since $\deg(c) = 0$

$\Rightarrow \deg(h_1(x)) + \deg(h_2(x)) \ldots \deg(h_s(x)) = 0$

$\Rightarrow \deg(h_1(x)) = 0, \deg(h_2(x)) = 0 \ldots \deg(h_s(x)) = 0$

$\Rightarrow h_1(x), h_2(x), \ldots h_s(x) \in R$

Thus the only factorization of c as an element of $R[x]$ are those it had as an element of R.

Also if $\alpha \in R$ is irreducible then $\alpha \in R[x]$ is also irreducible. Since R is UFD, therefore $c \in R$ can uniquely be expressed as product of irreducible elements of R and hence of $R[x]$

$\Rightarrow f(x) = c\,g(x)$ can uniquely be expressed as product of irreducible elements of $R[x]$

Hence, $R[x]$ is a UFD