

Let  $\Delta_1 = \{a = x_0, x_1, x_2, \dots, x_m = c\}$  be the partition of  $[a, c]$

and  $\Delta_2 = \{x_m = c, x_{m+1}, x_{m+2}, \dots, x_n = b\}$  be the partition of  $[c, b]$

so that  $\Delta = \Delta_1 \cup \Delta_2$  is the partition of  $[a, b]$  then, we have

$$s_{\Delta_1} + s_{\Delta_2} \leq s_{\Delta} \leq \sigma_{\Delta}$$

$$\Rightarrow \lim (s_{\Delta_1} + s_{\Delta_2}) \leq \lim s_{\Delta} \leq \lim \sigma_{\Delta}$$

$$\Rightarrow \lim s_{\Delta_1} + \lim s_{\Delta_2} \leq \lim s_{\Delta} \leq \lim \sigma_{\Delta}$$

$$\Rightarrow \int_a^c f(x) d\alpha(x) + \int_c^b f(x) d\alpha(x) \leq \int_a^b f(x) d\alpha(x) \quad \text{--- (1)}$$

Similarly,  $s_{\Delta_1} + s_{\Delta_2} \geq s_{\Delta} \geq \sigma_{\Delta}$  gives

$$\int_a^c f(x) d\alpha(x) + \int_c^b f(x) d\alpha(x) \geq \int_a^b f(x) d\alpha(x) \quad \text{--- (2)}$$

Combining (1) and (2) we get

$$\int_a^b f(x) d\alpha(x) = \int_a^c f(x) d\alpha(x) + \int_c^b f(x) d\alpha(x)$$

This prove the result (i)

Proof of (ii):-

$$\text{To prove } \left| \int_a^b f(x) d\alpha(x) \right| \leq \int_a^b |f(x)| d\alpha(x)$$

$$\leq \text{Max}_{a \leq x \leq b} |f(x)| \alpha(b)$$

Since,  $f(x)$  is continuous and  $\alpha(x)$  is of b.v. in  $[a, b]$

$$\Rightarrow \int_a^b f(x) d\alpha(x) \text{ exists}$$

Next,

$$\int_a^b f(x) d\alpha(x) = \lim_{S \rightarrow 0} \sum_{k=0}^{n-1} f(\xi_k) [\alpha(x_{k+1}) - \alpha(x_k)]$$

where  $\xi_k \in [x_k, x_{k+1}]$

$$\begin{aligned} \Rightarrow \left| \int_a^b f(x) d\alpha(x) \right| &= \lim_{S \rightarrow 0} \left| \sum_{k=0}^{n-1} f(\xi_k) [\alpha(x_{k+1}) - \alpha(x_k)] \right| \\ &\leq \lim_{S \rightarrow 0} \sum_{k=0}^{n-1} |f(\xi_k)| \cdot |\alpha(x_{k+1}) - \alpha(x_k)| \\ &= \int_a^b |f(x)| \cdot |d\alpha(x)| \\ &= \int_a^b |f(x)| \cdot |d\{\alpha(x) - \alpha(a)\}| \quad [\because \alpha(a) = 0] \\ &\leq \int_a^b |f(x)| dV_\alpha(x) \quad \left\{ \because |\alpha(x) - \alpha(a)| \leq V_\alpha(x) \right\} \\ \left| \int_a^b f(x) d\alpha(x) \right| &\leq \int_a^b |f(x)| dV_\alpha(x) \quad \text{--- (1)} \end{aligned}$$

Since  $f(x)$  is continuous in  $[a, b]$

$\Rightarrow f(x)$  is bounded there and it must attain its bound

i.e.  $|f(x)| \leq M, \forall x \in [a, b]$  and for some  $x \in [a, b]$   
 $|f(x)| \leq M$

$$\begin{aligned} \therefore \left| \int_a^b f(x) d\alpha(x) \right| &\leq M \int_a^b dV_\alpha(x) \quad [\text{by eqn (1)}] \\ &= M [V_\alpha(x)]_a^b \end{aligned}$$

$$\left| \int_a^b f(x) d\alpha(x) \right| \leq M [V_\alpha(b) - V_\alpha(a)] \quad \left\{ \because V_\alpha(a) = 0 \right\}$$

~~$$\left| \int_a^b f(x) d\alpha(x) \right| \leq M V_\alpha(b)$$~~

$$\left| \int_a^b f(x) d\alpha(x) \right| \leq \text{Max } |f(x)| V_\alpha(b)$$

Hence the theorem is proved



2. The existence of two integrals separately in  $\int_a^b f(x) d\alpha(x) = \int_a^c f(x) d\alpha(x) + \int_c^b f(x) d\alpha(x)$ , does imply the existence of the integral on left side. R.O. 79  
Justify your answer.

Ans Let us consider  $f(x)$  and  $\alpha(x)$  defined on  $[\frac{1}{2}, 2]$  such that

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for } 1 \leq x \leq 2 \end{cases}$$



$$\text{and } \alpha(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for } 1 \leq x \leq 2 \end{cases}$$

Now,  $\int_0^2 f(x) d\alpha(x) = \int_0^1 f(x) d\alpha(x) + \int_1^2 f(x) d\alpha(x) \quad \text{--- (A)}$

Now,  $\int_0^1 f(x) d\alpha(x) = \int_0^1 1 \cdot d\alpha(x) = [\alpha(x)]_0^1 = \alpha(1) - \alpha(0) = 0 - 1 = -1$

$\therefore \int_0^1 f(x) d\alpha(x)$  exists.

again,  $\int_1^2 f(x) d\alpha(x) = \int_1^2 0 \cdot d\alpha(x) = 0 \int_1^2 d\alpha(x) = 0$

$\therefore \int_1^2 f(x) d\alpha(x)$  exists.

Hence, integrals on R.H.S of (A) separately exist.

Now, we shall show that  $\int_0^2 f(x) d\alpha(x)$  does not exist.

$\because f(x)$  is discontinuous at  $x=1$ ,

$$\text{for, } f(1+0) = \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} 0 = 0$$

$$\text{and } f(1-0) = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} 1 = 1$$

$$\therefore f(1+0) \neq f(1-0)$$

$$\therefore \text{R.H.L.} \neq \text{L.H.L.}$$

Similarly,  $\alpha(x)$  is discontinuous at  $x=1$   
 then  $\int_0^2 f(x) d\alpha(x)$ , exists only when  $f(x)$  is  
 continuous and  $\alpha(x)$  is of b.v.

$\therefore \int_0^2 f(x) d\alpha(x)$  does not exist

Hence, the statement in the question is true  
 is established ✓

Theorem 5C If  $f(x)$  is continuous and  $\alpha(x)$  of b.v. in  $a \leq x \leq b$   
 then (1)  $F(x) = \int_a^x f(t) d\alpha(t)$ ;  $a \leq x \leq b$  is also of b.v.  
 in  $a \leq x \leq b$  and  $V_F(x) \leq \int_a^x |f(t)| \cdot |d\alpha(t)|$

Moreover

$$(2) F(x+) - F(x) = f(x) [\alpha(x+) - \alpha(x)]; \quad a \leq x \leq b$$

$$(3) F(x) - F(x-) = f(x) [\alpha(x) - \alpha(x-)]; \quad a \leq x \leq b.$$

Ans

If  $F(x)$  will be of b.v. in  $a \leq x \leq b$  iff  $V_F(b)$   
 exists, i.e.  $V_F(b)$  is bounded.

Let  $\Delta = \{a = x_0, x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n = b\}$ , be the  
 partition of  $[a, b]$ .

Now,

$$V_F(b) = \sup_0^{n-1} \sum |F(x_{k+1}) - F(x_k)|,$$

def. being extended to all partition of  $[a, b]$

$$= \sup_0^{n-1} \sum \left| \int_a^{x_{k+1}} f(t) d\alpha(t) - \int_a^{x_k} f(t) d\alpha(t) \right|$$

$$[\because \text{given } F(x) = \int_a^x f(t) d\alpha(t)]$$

$$= \sup_0^{n-1} \sum \left| \int_a^{x_{k+1}} f(t) d\alpha(t) + \int_{x_k}^a f(t) d\alpha(t) \right|$$

$$= \sup_0^{n-1} \sum \int_{x_k}^{x_{k+1}} |f(t) d\alpha(t)|$$

$$\leq \sup_0^{n-1} \sum \int_{x_k}^{x_{k+1}} |f(t)| dV\alpha(t) \quad \text{--- (A)}$$