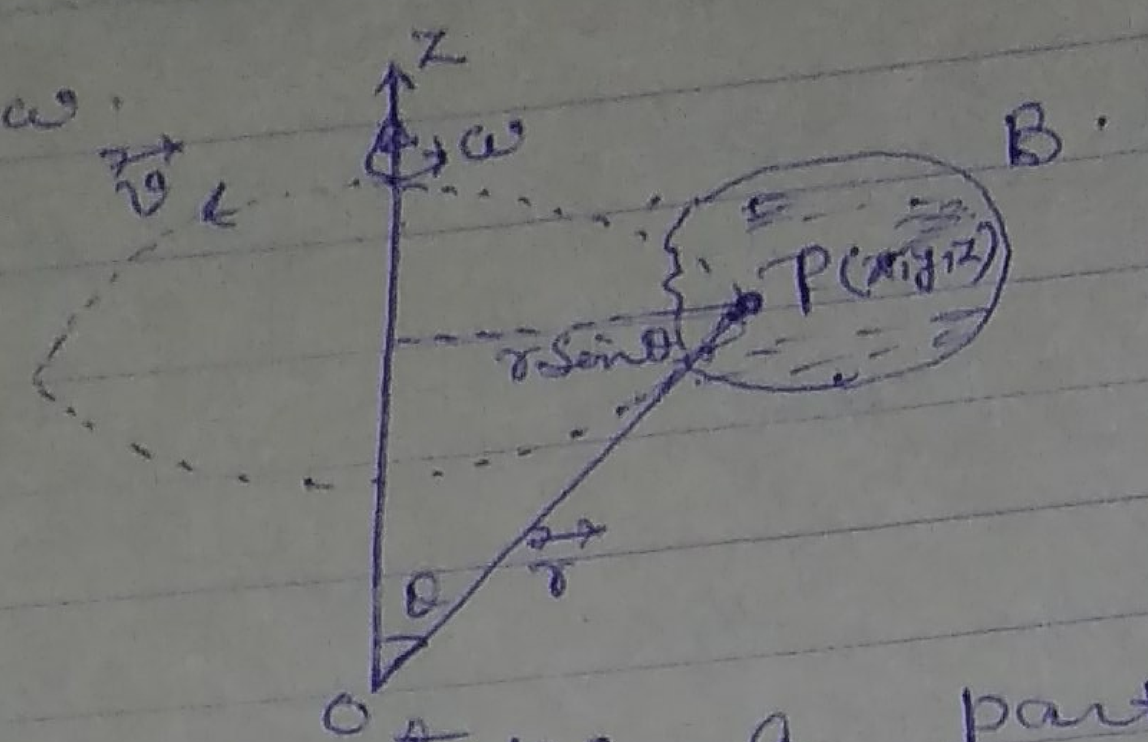
EXPRESSION FOR KINETIC ENERGY OF A ROTATING BODY :→

Let us consider a rigid body of B rotating about an axes ~~of~~ OZ with an angular velocity ω.



Now let us take a particle $P(x,y,z)$ of mass m, having position vector $\vec{r}$ with respect to the point O where $\angle POZ = \theta$, The linear velocity of the particle P will be given by,

$$\vec{v} = \omega r \sin\theta$$
$$= \vec{\omega} \times \vec{r}$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}$$
$$= \vec{i}\,[\omega_y z - \omega_z y] + \vec{j}\,[\omega_z x - \omega_x z] + \vec{k}\,[\omega_x y - \omega_y x].$$

$$\therefore v^2 = [\omega_y z - \omega_z y]^2 + [\omega_z x - \omega_x z]^2 + [\omega_x y - \omega_y x]^2$$
$$= \omega_x^2 [y^2 + z^2] + \omega_y^2 [z^2 + x^2] + \omega_z^2 [x^2 + y^2] - 2\omega_x \omega_y xy - 2\omega_y \omega_z yz - 2\omega_z \omega_x zx.$$

The kinetic energy of the particle

$$= \frac{1}{2} m v^2.$$

Hence the kinetic energy of the whole body, $T = \Sigma \frac{1}{2} m v^2$

$$= \frac{1}{2}\Big[\Sigma m(y^2+z^2)\,\omega_x^2 + \Sigma m(z^2+x^2)\,\omega_y^2 + \Sigma m(x^2+y^2)\,\omega_z^2 - 2\Sigma mxy\,\omega_x \omega_y - 2\Sigma myz\,\omega_y \omega_z - 2\Sigma mzx\,\omega_z \omega_x\Big].$$

$$\boxed{T = \frac{1}{2} I_{xx}\omega_x^2 + \frac{1}{2} I_{yy}\omega_y^2 + \frac{1}{2} I_{zz}\omega_z^2 - I_{xy}\omega_x\omega_y - I_{yz}\omega_y\omega_z - I_{zx}\cdot\omega_z\cdot\omega_x}$$

Where,

$$\left.\begin{aligned} I_{xx} &= \sum m(y^2+z^2) \\ I_{yy} &= \sum m(z^2+x^2) \\ I_{zz} &= \sum m(x^2+y^2) \end{aligned}\right\}$$

are the moments of inertia of the body about OX, OY and OZ respectively.

And,

$$\left.\begin{aligned} I_{xy} &= \sum mxy \\ I_{yz} &= \sum myz \\ I_{zx} &= \sum mzx \end{aligned}\right\}$$

Are the products of inertia of the body w.r.to the co-ordinate taken in pair.

The above equation represents the general expression for the kinetic energy of a rotating body.

Special case :→ If the axis X, Y, Z are taken as principal axis of the inertia, Then we will have,

$$I_{xy} = I_{yz} = I_{zx} = 0$$

& Hence,

$$\boxed{T = \frac{1}{2} I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2}$$

EXPRESSION