

# FOURIER TRANSFORM (5)

blem (4): Find the fourier transform of  $f(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

Q Given that  $f(x) = \begin{cases} 1-x^2, & -1 \leq x \leq 1 \\ 0, & |x| > 1 \end{cases}$

$$\begin{aligned}
 F\{f(x)\} &= \int_{-\infty}^{\infty} e^{-isx} f(x) dx \\
 &= \int_{-\infty}^{-1} e^{-isx} f(x) dx + \int_{-1}^1 e^{-isx} f(x) dx + \int_1^{\infty} e^{-isx} f(x) dx \\
 &\quad \text{put } x = -y \\
 &= \int_{\infty}^{-1} e^{isy} f(-y) (-dy) + \int_{-1}^1 e^{-isx} (1-x^2) dx + \int_1^{\infty} e^{-isx} \cdot 0 dx \\
 &= \int_1^{\infty} e^{isy} f(-y) dy + \int_{-1}^1 e^{-isx} (1-x^2) dx + 0 \\
 &= 0 + \int_{-1}^1 e^{-isx} (1-x^2) dx = \int_{-1}^1 e^{-isx} (1-x^2) dx \\
 &= \left[ (1-x^2) \frac{e^{-isx}}{-is} \right]_{-1}^1 - \int_{-1}^1 (-2x) \left[ \frac{e^{-isx}}{-is} \right] dx \\
 &= 0 - \frac{2}{is} \int_{-1}^1 x e^{-isx} dx \\
 &= -\frac{2}{is} \left[ x \frac{e^{-isx}}{-is} - \int 1 \cdot \frac{e^{-isx}}{-is} dx \right]_{-1}^1 \\
 &= -\frac{2}{is} \left[ \frac{x e^{-isx}}{-is} + \frac{1}{is} \frac{e^{-isx}}{-is} \right]_{-1}^1 \\
 &= -\frac{2}{is} \left[ \frac{e^{-isx}}{-is} + \frac{e^{-isx}}{-is} + \frac{1}{is} \left( \frac{e^{-isx}}{-is} - \frac{e^{-isx}}{-is} \right) \right]
 \end{aligned}$$



$$= \frac{2i^2}{is} \left[ \frac{e^{is} + e^{-is}}{-is} + \frac{e^{is} - e^{-is}}{s^2} \right]$$

$$= \frac{2i}{s} \left[ \frac{e^{is} + e^{-is}}{2} \times \frac{2}{-is} - \frac{e^{is} - e^{-is}}{2i} \times \frac{2i}{s^2} \right]$$

$$= \frac{2i}{s} \left[ \cos s \times \frac{-2}{is} - \sin s \times \frac{2i}{s^2} \right]$$

$$= \frac{2i}{s} \left[ \cos s \times \frac{2i}{s} - \frac{2i}{s^2} \sin s \right]$$

$$= \frac{4i^2}{s^3} [s^2 \cos s - \sin s]$$

$$= -\frac{4}{s^3} (s^2 \cos s - \sin s) = \frac{4}{s^3} (-s^2 \cos s + \sin s)$$

Problem (5) Find Fourier sine transform of  $e^{-|x|}$ . Ans

Hence evaluate  $\int_0^\infty \frac{x \sin(mx)}{1+x^2} dx$

Ans:  $\because |x| = x$  in  $(0, \infty)$ .

$$\text{Now } f_s\{F(x)\} = f_s(s) = \int_0^\infty F(x) \cdot \sin(sx) dx$$

$$= \int_0^\infty e^{-|x|} \sin(sx) dx = \int_0^\infty e^{-x} \sin(sx) dx$$

$$= \frac{s}{1+s^2} \quad \text{Ans} \quad \left( \because \int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2+b^2} \right)$$

Now for second part:

$$F(x) = f_s^{-1}\{f_s(s)\} = \frac{2}{\pi} \int_0^\infty f_s(s) \cdot \sin(sx) ds$$

$$\Rightarrow F(x) = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin(sx) ds$$



$$\Rightarrow \frac{\Lambda}{2} F(x) = \int_0^{\infty} \frac{s}{1+s^2} \sin(sx) ds$$

$$\Rightarrow \frac{\Lambda}{2} e^{-x} = \int_0^{\infty} \frac{s}{1+s^2} \sin(sx) ds$$

Replacing  $x$  by  $m$ . We have

$$\frac{\Lambda}{2} e^{-m} = \int_0^{\infty} \frac{s}{1+s^2} \sin(sm) ds$$

$$\Rightarrow \int_0^{\infty} \left( \frac{x}{1+x^2} \right) \sin(mx) dx = \frac{\Lambda}{2} e^{-m}$$

Problem (6) Find the Fourier transform of  $f(x) = e^{-a|x|}$ , where  $a > 0$  and  $x \in (-\infty, \infty)$ . Ans

Ans:  $\because |x| = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$

$$\begin{aligned} F\{f(x)\} = f(s) &= \int_{-\infty}^{\infty} f(x) e^{-isx} dx = \int_{-\infty}^{\infty} e^{-a|x|} e^{-isx} dx \\ &= \int_{-\infty}^0 e^{-a|x|} e^{-isx} dx + \int_0^{\infty} e^{-a|x|} e^{-isx} dx \\ &= \int_{-\infty}^0 e^{ax} e^{-isx} dx + \int_0^{\infty} e^{-ax} e^{-isx} dx \\ &= \int_{-\infty}^0 e^{(a-is)x} dx + \int_0^{\infty} e^{-(a+is)x} dx \end{aligned}$$



$$\begin{aligned}
 &= \left[ \frac{e^{(a-is)x}}{a-is} \right]_{x=-\infty}^0 + \left[ \frac{e^{-(a+is)x}}{-(a+is)} \right]_{x=0}^{\infty} \\
 &= \left[ \frac{1}{a-is} - \frac{0}{a-is} \right] - \left[ \frac{0}{a+is} - \frac{1}{a+is} \right] \\
 &= \frac{1}{a-is} + \frac{1}{a+is} = \frac{2a}{a^2+s^2} \quad \underline{\text{Ans}}
 \end{aligned}$$

Problem (7): Find the F.T. of  $f(x) = \begin{cases} x, & |x| \leq a \\ 0, & |x| > a \end{cases}$

Ans: Given that  $f(x) = \begin{cases} x, & -a \leq x \leq a \\ 0, & |x| > a \end{cases}$

Now  $F\{f(t)\} = \int_{-\infty}^{\infty} e^{-ist} f(t) dt.$

$$= \int_{-\infty}^{-a} e^{-ist} f(t) dt + \int_{-a}^a e^{-ist} f(t) dt + \int_a^{\infty} e^{-ist} f(t) dt.$$

put  $t = -y$

$$= \int_{\infty}^a e^{isy} f(-y) d(-y) + \int_{-a}^a e^{-ist} f(t) dt + \int_a^{\infty} e^{-ist} f(t) dt$$

$$= \int_a^{\infty} e^{isy} \cdot 0 dy + \int_{-a}^a e^{-ist} \cdot t dt + \int_a^{\infty} e^{-ist} \cdot 0 dt$$

$$= 0 + \left( \frac{e^{-ist}}{-is} \cdot t \right)_{-a}^a - \int_{-a}^a \frac{e^{-ist}}{-is} \cdot 1 \cdot dt$$

$$= \left( \frac{a}{-is} \right) (e^{-isa} + e^{isa}) + \frac{1}{is} \left( \frac{e^{-ist}}{-is} \right)_{-a}^a$$

$$= \frac{ia}{s} 2 \cos(sa) + \frac{1}{s^2} (e^{-isa} - e^{isa})$$

$$= \frac{2ias}{s^2} \cos(sa) - \frac{2i}{s^2} \left( \frac{e^{isa} - e^{-isa}}{2i} \right)$$

$$= \frac{2i}{s^2} [as \cdot \cos(sa) - \sin(sa)] \quad \underline{\text{Ans.}}$$