Example: Prove that the polynomial $1 + x + \cdots + x^{p-1}$ where p is prime number, is irreducible over field of rational numbers.

Sol^n Let $f(x) = 1 + x + \cdots + x^{p-1}$ ——— (1)

Multiplying both sides with $x-1$, we get

$$(x-1) f(x) = (x-1)(x^{p-1} + x^{p-2} + \cdots + x + 1)$$

$$\Rightarrow (x-1) f(x) = x^p - 1$$

Putting $x - 1 = y$ i.e. $x = y + 1$ we get

$$y f(y+1) = (y+1)^p - 1$$

$$\Rightarrow y f(y+1) = y^p + {}^pC_1 y^{p-1} + {}^pC_2 y^{p-2} + \cdots + {}^pC_{p-1} y + 1 - 1$$

$$= y^p + {}^pC_1 y^{p-1} + {}^pC_2 y^{p-2} + \cdots + {}^pC_{p-1} y$$

$$= y\left[y^{p-1} + {}^pC_1 y^{p-2} + {}^pC_2 y^{p-3} + \cdots + {}^pC_{p-1}\right]$$

$$\Rightarrow f(y+1) = y^{p-1} + {}^pC_1 y^{p-2} + {}^pC_3 y^{p-3} \cdots + {}^pC_{p-1} \quad \text{——— (2)}$$

Since ${}^pC_r = \dfrac{p(p-1)(p-2)\cdots(p-r+1)}{\lfloor r}$ $\quad 1 \leq r \leq p-1$

$\Rightarrow {}^pC_r$ is divisible by p for each $1 \leq r \leq p-1$

Now from ②, $f(y+1)$ is a polynomial with integer coefficients, Also p is prime number such that p divides each of the coefficients of $f(y+1)$ except the coefficient of y^{p-1}, which is 1. Also p^2 does not divide coefficient of constant term which is ${}^{p}C_{p-1} = p$. Therefore by Einstein's criterion for irreducibility, $f(y+1)$ is irreducible over field of rational numbers

$\Rightarrow f(x)$ is irreducible over field of rational numbers as $y+1 = x$

Ex: Show that the polynomial $x^4+x^3+x^2+x+1$ is irreducible over field of rational numbers

(Proceed as in previous problem)

Ex: Let R is UFD., then show that every prime element in R generates a prime ideal.

Solⁿ Let p be a prime element of a UFD "R"

Let $S = (p)$. be an ideal of R generated by p

We want to show that S is prime ideal

Let $a.b$ be any element of S where $a, b \in R$

$\Rightarrow ab = kp$ for some $k \in R$

$\Rightarrow p \mid ab$

$\Rightarrow p \mid a$ or $p \mid b$ ($\because p$ is prime element)

$\Rightarrow a = ps$ or $b = pt$ for some $s, t \in R$

$\Rightarrow a \in (p)$ or $b \in (p)$

$\Rightarrow (p)$ is prime ideal