Closed Set

A subset F of a metric space X is said to be closed if its complement F^c is open.

Th^m If (X, d) be a metric space then
(i) X, and ϕ are closed sets.
(ii) Arbitrary intersection of closed sets is closed.
(iii) Finite union of closed sets is closed.

Proof (i) Since X is open set
$\Rightarrow X^c$ is closed set (By def^n)
$\Rightarrow \phi$ is closed set
Also, since ϕ is open set
$\Rightarrow \phi^c$ is closed set
$\Rightarrow X$ is open set.

(ii) Let $\{F_\lambda : \lambda \in \Delta\}$ be arbitrary family of closed sets in a metric space X.
Since F_λ is closed set $\forall \lambda \in \Delta$
$\Rightarrow F_\lambda^c$ is open $\forall \lambda \in \Delta$
$\Rightarrow \bigcup_{\lambda \in \Delta} F_\lambda^c$ is open ($\because$ Arbitrary union of open sets is open)

$\Rightarrow \left\{\bigcap_{\lambda \in \Delta} F_\lambda\right\}^c$ is open set (By De Morgen's law)

$\Rightarrow \bigcap_{\lambda \in \Delta} F_\lambda$ is closed set

(Proved)

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(iii) Let $\{F_1, F_2, \ldots F_m\}$ be a finite family of closed subsets of a metric space X.

Since F_i is closed set $\because i = 1, 2, 3, \ldots m$

$\Rightarrow F_i^c$ is open set. $i = 1, 2, 3, \ldots m$

$\Rightarrow \bigcap_{i=1}^{m} F_i^c$ is open set ($\because$ Finite intersection of open sets is open)

$\Rightarrow \left(\bigcup_{i=1}^{m} F_i\right)^c$ is open set

$= \bigcup_{i=1}^{m} F_i$ is closed set

(Proved)

Limit point:
Let F be a subset of a metric space X. A point $x \in X$ is said to be a limit point of F if every open sphere centred at x contains at least one point of F other than x.

It is also called Accumulation point or cluster point.

Derived set: If F is a subset of a metric space X, then set of all limit points of F is called Derived set of F. It is denoted by $D(F)$ or F'

Closure of a set: If F is a subset of a metric space X, then closure of F is $F \cup D(A)$ or $F \cup F'$.
It is denoted by $\bar{F}$ or $cl(F)$

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Thm A subset of a metric space is closed if and only if it contains all of its limit points.

Proof. Let F be a closed subset of a metric space. Let α be a limit point of F. We want show that $\alpha \in F$

Let us assume to the contradiction that $\alpha \notin F$.

$\Rightarrow$ $\alpha \in F^c$ and also F^c is open

$\Rightarrow$ $\exists\, r>0$ such that

$$S_r(\alpha) \subseteq F^c$$

ie $S_r(\alpha)$ contains no point of F

$\Rightarrow$ α cannot be a limit point of F

Which is a contradiction, Hence $\alpha \in F$

ie F contains all its limit points.

Conversely Let F is a subset of metric space X which contains all its limit points. We want to show that F is closed. ie equivalently F^c is open

Let $x \in F^c$

$\Rightarrow$ $x \notin F$

$\Rightarrow$ x is not a limit point of F

$\Rightarrow$ $\exists\, r>0$ such that $S_r(x)$ does not contain any point of F.

$\Rightarrow$ $S_r(x) \cap F = \phi$

$\Rightarrow$ $S_r(x) \subseteq F^c$

$\Rightarrow$ F^c is open

(Proved)

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Thm Let F be any subset of metric space X then the derived set F' is closed set.

Proof We want to show that derived set F' is closed ie to show that F' contains all its limit points.

Let x be a limit point of F'. We will show that $x \in F'$.

Let $S_r(x)$ be any open sphere centred at x

Since 'x is limit point of F', $\exists$ $y \in F'$ such that $y \in S_r(x)$ and $y \neq x$

$$\Rightarrow d(x,y) < r \longrightarrow ①$$

Let $r_1 = r - d(x,y)$. Obviously $r_1 > 0$

Consider open sphere $S_{r_1}(y)$

Let $z \in S_{r_1}(y)$. $z \neq y \neq x$ and $z \in F$ ($\because$ y is limit pt of F)

$$\Rightarrow d(z,y) < r_1 \quad — ②$$

Now, $d(z,x) \leq d(z,y) + d(y,x)$

$$< r_1 + (r - r_1) = r$$

$$\Rightarrow z \in S_r(x)$$

$\Rightarrow$ Every open sphere $S_r(x)$ contains a point of F other than x.

$\Rightarrow$ x is limit point of F

$\Rightarrow$ $x \in F'$

$\Rightarrow$ F' is closed.

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