Since $f(x)$ is continuous in $[a,b]$

$\Rightarrow f(x)$ is continuous in each of sub-interval in $[a,b]$

$\Rightarrow f(x)$ is continuous in $[x_k, x_{k+1}]$

$\Rightarrow f(x)$ is bounded in $[x_k, x_{k+1}]$

$\Rightarrow f(x) \le M \quad \forall x \in [x_k, x_{k+1}]$

$\therefore$ from eqn (A), we have

$$V_F(b) \le \sup \sum_0^{n-1} \int_{x_k}^{x_{k+1}} |f(t)|\, dV_\alpha(t)$$

$$\le \sup \sum_0^{n-1} M \int_{x_k}^{x_{k+1}} dV_\alpha(t)$$

$$= M \sup \sum_0^{n-1} \int_{x_k}^{x_{k+1}} dV_\alpha(t)$$

$$= M \sup \int_a^b dV_\alpha(t)$$

$$= M \sup [V_\alpha(t)]_a^b$$

$$\Rightarrow V_F(b) = M \sup [V_\alpha(b) - V_\alpha(a)]$$

$$\Rightarrow V_F(b) = M V_\alpha(b) \qquad \{\because V_\alpha(a) = 0\}$$

As, $\alpha(x)$ is of b.v. ~~of~~ in $[a,b] \Rightarrow V_\alpha(b)$ exists

So, $MV_\alpha(b)$ exists $=$ a finite quantity say L

and hence $V_F(b) \le L$

$\Rightarrow V_F(b)$ exists

$\Rightarrow F(x)$ is of b.v. in $[a,b]$.

2nd part:-

To prove $V_F(x) \le \int_a^x |f(t)| \cdot |d\alpha(t)|$

we have

$$V_F(x) = \sup \sum_0^{n-1} |F(x_{k+1}) - F(x_k)|$$

let $\Delta = \{x_0 = a, x_1, \ldots x_n = b\}$ is a partition of $[a, b$

$$= \sup \sum_0^{n-1} \int_a^{x_{k+1}} f(t)\, d\alpha(t) - \int_a^{x_k} f(t)\, d\alpha(t) |$$

Not: $F(x+) = F(x+h)$
$F(x-) = F(x-h)$

$$= \sup \sum_{0}^{n-1} \left| \int_{a}^{x_{k+1}} f(t)\, d\alpha(t) + \int_{x_k}^{a} f(t)\, d\alpha(t) \right|$$

$$= \sup \sum_{0}^{n-1} \left| \int_{x_k}^{x_{k+1}} f(t)\, d\alpha(t) \right|$$

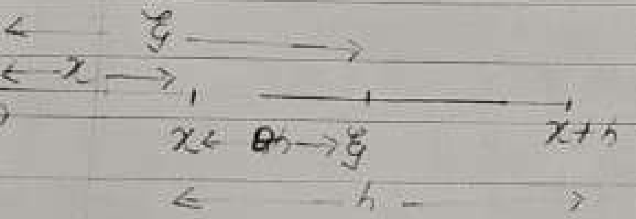
$$\leq \sup \sum_{0}^{n-1} \int_{x_k}^{x_{k+1}} |f(t)| \cdot |d\alpha(t)|$$

$$= \int_{a}^{x} |f(t)| \cdot |d\alpha(t)|$$

$$\Rightarrow V_F(x) \leq \int_{a}^{x} |f(t)| \cdot |d\alpha(t)|$$

<u>Proof of (2)</u> :-

We have, $F(x^+) - F(x) = \lim_{h \to 0} F(x+h) - F(x)$



$$= \lim_{h \to 0} \{ F(x+h) - F(x) \}$$

$$= \lim_{h \to 0} \left\{ \int_{a}^{x+h} f(t)\, d\alpha(t) - \int_{a}^{x} f(t)\, d\alpha(t) \right\}$$

$$= \lim_{h \to 0} \left\{ \int_{a}^{x+h} f(t)\, d\alpha(t) + \int_{x}^{a} f(t)\, d\alpha(t) \right\}$$

$$= \lim_{h \to 0} \int_{x}^{x+h} f(t)\, d\alpha(t)$$

By Apply Mean value theorem

$$= \lim_{h \to 0} f(\xi) [\alpha(x+h) - \alpha(x)]$$

where $\xi \in [x, x+h]$

$$= \lim_{h \to 0} f(x+\theta h) [\alpha(x+h) - \alpha(x)]$$

Taking $\xi = (x+\theta h)$, $0 < \theta < 1$

$$\Rightarrow F(x+) - F(x) = f(x) [\alpha(x+) - \alpha(x)], \quad a \leq x \leq b$$

[$\because$ $f(x)$ is constant so $\lim_{h \to 0} f(x+\theta h) = f(x+) - f(x)$]

Proof of 20

We have $F(x) - F(x-) = F(x) - \lim_{h \to 0} F(x-h)$

$$= \lim_{h \to 0} [F(x) - F(x-h)]$$

$$= \lim_{h \to 0} \left\{ \int_a^x f(t)\,d\alpha(t) - \int_a^{x-h} f(t)\,d\alpha(t) \right\}$$

$$= \lim_{h \to 0} \int_{x-h}^{x} f(t)\,d\alpha(t)$$

Applying Mean value theorem,

$$= \lim_{h \to 0} f(x-h+\theta h)[\alpha(x) - \alpha(x-h)] \quad 0<\theta<1$$

$$\Rightarrow F(x) - F(x-) = f(x)\,[\alpha(x) - \alpha(x-)]$$

VVI

Theorem 6(a) :- If $f(x)$ is continuous and $\varphi(x)$ belongs to $\mathcal{L}$ in $a \le x \le b$ and if $\alpha(x) = \int_c^x \varphi(t)\,dt$ $a \le c \le b$, $a \le x \le b$, then

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\varphi(x)\,dx = \int_a^b f(x)\,\alpha'(x)\,dx$$

Note : $\mathcal{L} \to$ Leibjes integral

Proof :- Since definite integral of a function is always of b.v., so we must say $\alpha(x)$ is of b.V. in $[a, b]$, where $\alpha(x) = \int_c^x \varphi(t)\,dt$

Hence, we have $f(x)$ is continuous and $\alpha(x)$ is of b.V. in $[a, b]$. So $\int_a^b f(x)\,d\alpha(x)$ exists

Thus, we can say

$$\int_a^b f(x)\,d\alpha(x) = \lim_{\delta \to 0} \sum_0^{n-1} f(\xi_k)\left[\alpha(x_{k+1}) - \alpha(x_k)\right]$$

where $\Delta = \{x_0 = a, x_1, x_2, \dots x_k, x_{k+1}, \dots x_n = b\}$ is a partition of $[a, b]$ and $\xi_k \in [x_k, x_{k+1}]$

$$= \lim_{\delta \to 0} \sigma_\Delta \quad , \text{ where } \sigma_\Delta = \sum_{0}^{n-1} f(\xi_k)[\alpha(x_{k+1}) - \alpha(x_k)]$$

Now, we shall prove

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\varphi(x)\,dx$$

i.e. $$\lim_{\delta \to 0} \sigma_\Delta - \int_a^b f(x)\varphi(x)\,dx = 0$$

$$\Rightarrow \lim_{\delta \to 0} \left[\sigma_\Delta - \int_a^b f(x)\varphi(x)\,dx\right] = 0$$

Now,

$$\left|\sigma_\Delta - \int_a^b f(x)\varphi(x)\,dx\right| = \left|\sum_{0}^{n-1} f(\xi_k)[\alpha(x_{k+1}) - \alpha(x_k)] - \int_a^b f(x)\varphi(x)\,dx\right|$$

$$= \left|\sum_{0}^{n-1} f(\xi_k)\left[\int_c^{x_{k+1}} \varphi(t)\,dt - \int_c^{x_k} \varphi(t)\,dt\right] - \int_a^b f(x)\varphi(x)\,dx\right|$$

$$\left|\sigma_\Delta - \int_a^b f(x)\varphi(x)\,dx\right| = \left|\sum_{0}^{n-1} f(\xi_k)\left[\int_c^{x_{k+1}} \varphi(t)\,dt + \int_{x_k}^{c} \varphi(t)\,dt\right] - \int_a^b f(x)\varphi(x)\,dx\right|$$

$$= \left|\sum_{0}^{n-1} f(\xi_k)\int_{x_k}^{x_{k+1}} \varphi(t)\,dt - \int_a^b f(t)\varphi(t)\,dt\right|$$

$$\left\{\because \int_a^b F(x)\,dx = \int_a^b F(y)\,dy\right\}$$

$$= \left|\sum_{0}^{n-1} f(\xi_k)\int_{x_k}^{x_{k+1}} \varphi(t)\,dt - \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} f(t)\varphi(t)\,dt\right|$$

$$= \left|\sum_{0}^{n-1} \int_{x_k}^{x_{k+1}} f(\xi_k)\varphi(t)\,dt - \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} f(t)\varphi(t)\,dt\right|$$

$$= \left|\sum_{0}^{n-1} \left\{\int_{x_k}^{x_{k+1}} [f(\xi_k) - f(t)]\,\varphi(t)\,dt\right\}\right|$$

$$\leq \sum_{0}^{n-1} \left|\int_{x_k}^{x_{k+1}} [f(\xi_k) - f(t)]\,\varphi(t)\,dt\right|$$

$$\leq \sum_{0}^{n-1} \int_{x_k}^{x_{k+1}} |f(\xi_k) - f(t)| \cdot |\varphi(t)|\,dt$$

$$< \sum_{0}^{n-1} \epsilon \int_{x_k}^{x_{k+1}} |\varphi(t)|\,dt$$

$\{\because f(x)$ is constant in $[a,b] \Rightarrow f(x)$ is constant at $x = \xi$ & hence $|f(\xi_k) - f(t)| < \epsilon,\ \epsilon > 0$, whenever $|\xi_k - t| < \delta, \eta > 0\}$

$$= \epsilon \sum_{0}^{n-1} \int_{x_k}^{x_{k+1}} |\varphi(t)|\,dt$$

$$= \epsilon \int_a^b |\varphi(t)|\,dt$$

$$= \epsilon \cdot A, \quad \text{for } \varphi(x) \in L \text{ in } [a,b] \Rightarrow \int_a^b |\varphi(x)|\,dx \text{ is fd}$$

$$= \epsilon_1 \text{ (say)} \qquad \Rightarrow \int_a^b |\varphi(x)|\,dx = A \text{ (say)}$$

$$\Rightarrow \left| \sigma_\Delta - \int_a^b f(x)\varphi(x)\,dx \right| < \epsilon_1, \quad \exists\ \epsilon_1 > 0$$

$$\Rightarrow \lim_{\delta \to 0} \left\{ \sigma_\Delta - \int_a^b f(x)\varphi(x)\,dx \right\} = 0$$

$$\Rightarrow \lim_{\delta \to 0} \sigma_\Delta - \int_a^b f(x)\varphi(x)\,dx = 0$$

$$\Rightarrow \lim_{\delta \to 0} \sigma_\Delta = \int_a^b f(x)\varphi(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\varphi(x)\,dx$$

Also, $\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'\,dx$ $\{\because \alpha$ is a funct. of $x\}$

$$\therefore \int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\varphi(x)\,dx = \int_a^b f(x)\alpha'(x)\,dx$$

Theorem 6 (b) :- If $f(x)$ and $\varphi(x)$ are continuous and $\varphi(x)$ is of b.v. in $a \le x \le b$ and $\beta(x) = \int_c^x \varphi(t)\,d\alpha(t)$ $(a \le x \le b,\ a \le c \le b)$ then $\int_a^b f(x)\,d\beta(x) = \int_a^b f(x)\varphi(x)\,d\alpha(x)$.

Proof :- Since $\varphi(x)$ is continuous and $\alpha(x)$ is of b.v. in $a \le x \le b$