FOURIER TRANSFORM (13)

Theorem (3) Parseval's identity for Fourier Transform OR, Rayleigh's Theorem: If $f(p)$ and $g(p)$ are complex Fourier transforms of $F(x)$ and $G(x)$ respectively, then

(i) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\,\overline{g(p)}\,dp = \int_{-\infty}^{\infty} F(x)\,\overline{G(x)}\,dx$$

(ii) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(p)|^2\,dp = \int_{-\infty}^{\infty} |F(x)|^2\,dx.$$

Where bar represents the complex conjugates.

Proof: Using the inversion formula for F.T., we have

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(p)\,e^{ipx}\,dp \quad \text{——— (1)}$$

Taking conjugate complex of both sides of (1),

$$\overline{G(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(p)}\,e^{-ipx}\,dp \quad \text{——— (2)}$$

$$\therefore \int_{-\infty}^{\infty} F(x)\,\overline{G(x)}\,dx = \int_{-\infty}^{\infty} F(x)\,dx \cdot \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(p)}\,e^{-ipx}\,dp\right\} \quad \text{by (2)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x)\,\overline{g(p)}\,e^{-ipx}\,dx\,dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x)\,\overline{g(p)}\,e^{-ipx}\,dp\,dx$$

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$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(p)}\,dp\left[\int_{-\infty}^{\infty} F(x)\,e^{-ipx}\,dx\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} F(x)\,\overline{G(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(p)}\,dp \cdot f(p)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\,\overline{g(p)}\,dp \quad \text{--- (3)}$$

This proves the first part.

Putting $G(x) = F(x)$ in (3), we have

$$\int_{-\infty}^{\infty} F(x)\,\overline{F(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\,\overline{f(p)}\,dp$$

$$\Rightarrow \int_{-\infty}^{\infty} |F(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(p)|^2\,dp$$

Proved.

Note: This theorem is also called Planeterel's Theorem.

Theorem (4) Parseval's identity for Fourier series:

Statement: Suppose the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in the interval $-l < x < l$, then

$$\frac{1}{l}\int_{-l}^{l} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

when the integral on L.H.S exist.

Proof: Let the Fourier series of $f(x)$ converges uniformly to $f(x)$ at every point of the interval $-l < x < l$ so that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad (1)$$

and that term by term integration of this series is possible. Here

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\,dx \quad , n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\,dx \quad , n = 1, 2, 3, \ldots$$

Multiplying Eq (1) by $f(x)$ and integrating term by term from $-l$ to l, we have

$$\int_{-l}^{l} [f(x)]^2\,dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty}\int_{-l}^{l} f(x)\left[a_n\cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]dx$$

$$= \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty} a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx + \sum_{n=1}^{\infty} b_n\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{a_0}{2}\cdot l a_0 + \sum_{n=1}^{\infty} l\,(a_n^2 + b_n^2)$$

$$\Rightarrow \frac{1}{l}\int_{-l}^{l} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum (a_n^2 + b_n^2)$$

Proved

Problem (1) Find the complex form of the fourier integral representation of

$$f(x) = \begin{cases} e^{-kx} & , x > 0,\ k > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

Ans By complex (or, exponential) form of Fourier integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{-ist}\, e^{isx}\,ds\,dt.$$

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or $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\, ds \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\, ds \int_{0}^{\infty} e^{-kt}\, e^{-ist}\, dt$ [by def.]

But $\int_0^{\infty} e^{-kt}\cdot e^{-ist}\, dt = \int_0^{\infty} e^{-t(k+is)}\, dt$

$= \left[\frac{e^{-t(k+is)}}{-(k+is)}\right]_0^{\infty} = \frac{1}{k+is}$

Hence $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\, ds \cdot \frac{1}{k+is}$

Ans

Problem 2: Find fourier integral of the function

$$f(x) = \begin{cases} 0 & , x<0 \\ \frac{1}{2} & , x=0 \\ e^{-x} & , x>0 \end{cases}$$

Answer: Since $f(x)$ is an exponential form, Then by F. exponential formula

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{ist} \cdot e^{isx}\, ds\, dt$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} I \cdot e^{isx}\, ds$ ——— (1)

Where $I = \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$

$= \int_{-\infty}^{0} f(t)\, e^{-ist}\, dt + \int_0^{\infty} f(t)\, e^{-ist}\, dt$

$$I = \int_{-\infty}^{0} 0 \cdot e^{-ist}\, dt + \int_{0}^{\infty} e^{-t} \cdot e^{-ist}\, dt$$

$$= 0 + \int_{0}^{\infty} e^{-(1+is)t}\, dt = \int_{0}^{\infty} e^{-pt}\, dt, \quad \text{where } p = 1+is$$

$$= \left\{ \frac{e^{-pt}}{-p} \right\}_{t=0}^{\infty} = -\frac{1}{p}(0-1) = \frac{1}{p} = \frac{1}{1+is} = \frac{1-is}{1+s^2}$$

Now by ①

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1-is}{1+s^2} \right) e^{isx}\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{1+s^2} \right) (1-is) \{ \cos(sx) + i \sin(sx) \} ds$$

using $\int_{-a}^{a} f(x)\, dx = \begin{cases} 0 & \text{if } f(-x) = -f(x) \\ 2\int_{0}^{a} f(x)\, dx, & \text{if } f(-x) = f(x) \end{cases}$

Then $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{1+s^2} \right) [\cos sx + i \sin sx - is \cos sx + s \sin sx]\, ds$$

$$= \frac{2}{2\pi} \int_{0}^{\infty} \left(\frac{1}{1+s^2} \right) (\cos sx + 0 - 0 + s \cdot \sin sx)\, ds$$

$$= \frac{1}{\pi} \int_{0}^{\infty} [\cos sx + s \cdot \sin sx] \frac{ds}{1+s^2}$$

This is the required fourier integral of $f(x)$.

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Problem (3) Using fourier sine integral formula, Prove that

$$\int_0^\infty \left\{ \frac{1-\cos(\pi\lambda)}{\lambda} \right\} \sin(x\lambda)\, d\lambda = \begin{cases} \frac{\pi}{2}, & 0<x<\pi \\ 0, & x>\pi \end{cases}$$

Answer Let $f(x) = \begin{cases} \frac{\pi}{2}, & 0<x<\pi \\ 0, & x>\pi \end{cases}$ —— (a)

By fourier sine integral formula

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \sin sx \, ds\, dt$$

$$= \frac{2}{\pi} \int_0^\infty \sin(sx)\, ds \cdot \int_0^\infty f(t) \sin(st)\, dt \quad \text{—— (1)}$$

According to (a)

$$\int_0^\infty f(t) \sin(st)\, dt = \int_0^\pi \frac{\pi}{2} \cdot \sin(st)\, dt$$

$$= -\frac{\pi}{2s} \left\{ \cos(st) \right\}_{t=0}^{\pi}$$

$$= \frac{\pi}{2s} \left\{ 1 - \cos(\pi s) \right\}$$

Now by (1)

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin(sx)\, ds \cdot (1-\cos\pi s) \frac{\pi}{2s}.$$

Replacing s by λ, we get

$$\int_0^\infty \left\{ \frac{1-\cos(\pi\lambda)}{\lambda} \right\} \sin(x\lambda)\, d\lambda = f(x) = \begin{cases} \frac{\pi}{2}, & 0<x<\pi. \\ 0, & x>\pi \end{cases}$$

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