PRINCIPLE OF LEAST ACTION

STATEMENT :→ The Principle of least action States that a System in moving from one Configuration State (at time t_1) to another configuration State (at time t_2) follows a Path along which the time integral of twice the Kinetic energy of the System, i.e. the action of the System is the least, Provided the Hamiltonian of the System is independent of time.

Mathematically,

$$\int_{t_1}^{t_2} 2T\,dt = \text{Action } A = \text{The least}$$

$$\text{or, } \Delta \int_{t_1}^{t_2} 2T\,dt = 0$$

where Δ represents a new type of variation of the Path which allows time as well as position co-ordinate (q_k) to vary.

DEDUCTION OF THE PRINCIPLE :→

In the Δ variation, t varies even at the end points, where the variation of q_k is zero, While in δ variation t remains constant. If α is a variational Parameter, then the Δ variation is mathematically defined as;

$$\Delta q = \left[\frac{dq}{d\alpha}\right] d\alpha \quad \text{Where as, } \delta q = \left(\frac{\partial q}{\partial \alpha}\right) d\alpha.$$

$$\text{or, } \Delta q = d\alpha \left[\frac{\partial q}{\partial \alpha} + \dot{q}\frac{dt}{d\alpha}\right] \text{-----}$$

As $q = q(\alpha, t)$

or, $\Delta q = d\alpha \left[\frac{\partial q}{\partial \alpha}\right] + \dot{q}\, d\alpha \left[\frac{dt}{d\alpha}\right]$

or, $\Delta q = \delta q + \dot{q}\Delta t$ ...... (1)

By the definition, the action A is given by,

$$A = \int_{t_1}^{t_2} 2T\, dt$$

$$= \int_{t_1}^{t_2} (L+H)\, dt \quad \text{---- As } L = T - V \text{ \& } H = T + V,\ L + H = 2T$$

$$= \int_{t_1}^{t_2} L\,dt + \int_{t_1}^{t_2} H\,dt.$$

$$= \int_{t_1}^{t_2} L\,dt + H\left[t_2 - t_1\right] \quad \text{------ If H is taken to be Constant.}$$

$$\therefore \Delta A = \Delta \int_{t_1}^{t_2} L\,dt + H\left[\Delta t_2 - \Delta t_1\right] \quad \text{----- (2)}$$

Value of $\Delta \int_{t_1}^{t_2} L\,dt$:→

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Let, $\int_{t_1}^{t_2} L\cdot dt = I(t_2) - I(t_1)$ ---- when $\dot{I} = L$

$$\therefore \Delta \int_{t_1}^{t_2} L\cdot dt = \Delta I(t_2) - \Delta I(t_1)$$

$$= \left[\delta I(t_2) + \dot{I}(t_2)\Delta t_2\right] - \left[\delta I(t_1) + \dot{I}(t_1)\Delta t_1\right]$$

$$= \delta\left[I(t_2) - I(t_1)\right] + \left[\dot{I}(t_2)\Delta t_2 - \dot{I}(t_1)\Delta t_1\right]$$

$$= \delta\int_{t_1}^{t_2} L\cdot dt + \left[L\Delta t\right]_{t_1}^{t_2} \quad \text{------} \quad (3.)$$

Further :→

$$\delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \delta L\,dt$$

$$= \int_{t_1}^{t_2}\sum_k \left[\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right]dt$$

---- As $L = L(q_k, \dot{q}_k)$

$$= \int_{t_1}^{t_2}\sum_k \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\delta q_k\right]dt$$

Using Langranges equation.

$$= \sum_k \int_{t_1}^{t_2} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\cdot\delta q_k\right]dt$$

$$= \sum_k \left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2}$$

$$= \sum_k \left[P_k(\Delta q_k - \Delta t\,\dot{q}_k)\right]_{t_1}^{t_2}$$

Using equation (1) and Putting $\frac{\partial L}{\partial \dot{q}_k} = P_k$

$$= \sum_k \left[P_k\Delta q_k - P_k\dot{q}_k\Delta t\right]_{t_1}^{t_2}$$

$$= \sum_k \left[-P_k\dot{q}_k\Delta t\right]_{t_1}^{t_2}$$

--- As at the end points in Δ- variation $\Delta q_k = 0$.

Putting this value in equation (3) we get,

$$\Delta \int_{t_1}^{t_2} L\,dt = \left[\left(-\sum_k p_k \dot{q}_k + L \right) \Delta t \right]_{t_1}^{t_2}$$

$$= \left[-\left(\sum_k p_k \dot{q}_k - L \right) \Delta t \right]_{t_1}^{t_2}$$

$$= \left[-H \Delta t \right]_{t_1}^{t_2} \qquad \Big| \text{ As } H = \sum_k p_k \dot{q}_k - L$$

$$= -H \left[\Delta t_2 - \Delta t_1 \right]$$

Hence from equation (2) we get,

$$\Delta A = 0$$

$$\Rightarrow \boxed{\Delta \int_{t_1}^{t_2} 2T\,dt = 0}$$

This proves the principle of Least action.