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FOURIER TRANSFORM (4)

Problem (8) Find the inverse F.T. of $\bar{f}(p) = e^{-|p|y}$, where $y \in (-\infty, \infty)$.

Ans: $$|p| = \begin{cases} -p, & p \le 0 \\ p, & p \ge 0 \end{cases}$$

$$F^{-1}\{\bar{f}(p)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(p)\, e^{ipx}\, dp$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} \bar{f}(p)\, e^{ipx}\, dp + \int_{0}^{\infty} \bar{f}(p)\, e^{ipx}\, dp\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-|p|y} \cdot e^{ipx}\, dp + \int_{0}^{\infty} e^{-|p|y} \cdot e^{ipx}\, dp\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{py} \cdot e^{ipx}\, dp + \int_{0}^{\infty} e^{-py} \cdot e^{ipx}\, dp\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(y+ix)p}\, dp + \int_{0}^{\infty} e^{-(y-ix)p}\, dp\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{e^{(y+ix)p}}{y+ix}\right\}_{-\infty}^{0} + \left\{\frac{e^{-(y-ix)p}}{-(y-ix)}\right\}_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{y+ix}(1-0) + \frac{1}{ix-y}(0-1)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{ix+y} - \frac{1}{ix-y}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\cancel{ix}-y-\cancel{ix}-y}{-x^2-y^2}\right] = \frac{y}{\pi(x^2+y^2)}$$ Ans

Problem (9) Prove that the fourier transform of the step function $f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2l}, & -l < x < l \\ 0, & |x| > l \end{cases}$

is $\frac{\sqrt{2\pi}}{2l} \sin sl$.

Answer $F\{F(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{-isx}\, dx$

$$= \int_{-\infty}^{-l} F(x)\, e^{-isx}\, dx + \int_{-l}^{l} F(x) . e^{-isx}\, dx + \int_{l}^{\infty} F(x)\, e^{-isx}\, dx$$

Put $x = -y$

$$= \int_{+\infty}^{+l} F(-y)\, e^{isy} (-dy) + \int_{-l}^{l} \frac{\sqrt{2\pi}}{2l} . e^{-isx}\, dx + \int_{l}^{\infty} 0 . e^{-isx}\, dx$$

$$= \int_{l}^{\infty} 0 . e^{isy}\, dy + \frac{\sqrt{2\pi}}{2l} . \left[\frac{e^{-isx}}{-is}\right]_{x=-l}^{x=l} + 0$$

$$= 0 + \frac{\sqrt{2\pi}}{2l} \left[\frac{e^{-isl} - e^{+isl}}{-is}\right] + 0$$

$$= \frac{\sqrt{2\pi}}{2l} \left[\frac{e^{isl} - e^{-isl}}{2i} \times \frac{2i}{is}\right]$$

$$= \frac{\sqrt{2\pi}}{2l} . \sin sl \times \frac{2}{s} = \frac{\sqrt{2\pi}}{sl} . \sin(sl)$$ Ans

Problem (10) Solve the integral equation

$$\int_0^{\infty} f(x) \cos \lambda x\, dx = e^{-\lambda}.$$

Ans: $\because \int_0^{\infty} f(x) \cos sx\, dx = e^{-s}$ ——— (1)

By definition. $F_c\{f(x)\} = \int_0^{\infty} f(x) \cos sx\, dx = f_c(s)$ ——— (2)

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and $F_c^{-1}\{\bar{f}_c(s)\} = f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}_c(s)\cos sx\,dx$ — (3)

By (1) & (2), we have

$$\bar{f}_c(s) = e^{-s}.$$

Now by (3), we have

$$f(x) = \frac{2}{\pi}\int_0^\infty e^{-s}\cos sx\,dx$$

$$= \frac{2}{\pi}\cdot\frac{1}{1+x^2}\left(\because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}\right).$$

$$\therefore f(x) = \frac{2}{\pi(1+x^2)} \quad \text{Ans}$$

Problem (11) Solve the integral equation

$$\int_0^\infty F(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda, & \text{for } 0 \le \lambda < 1 \\ 0, & \text{for } \lambda > 1 \end{cases}$$

Answer By definition

$$F_c\{F(x)\} = \int_0^\infty F(x)\cos\lambda x\,dx = f_c(\lambda) \quad \text{— (1)}$$

Then $f_c(\lambda) = \begin{cases} 1-\lambda, & \text{for } 0 \le \lambda \le 1 \\ 0, & \text{for } \lambda > 1 \end{cases}$

The inverse formula relative to (1) is

$$F_c^{-1}\{f_c(\lambda)\} = F(x) = \frac{2}{\pi}\int_0^\infty f_c(\lambda)\cos\lambda x\,d\lambda$$

From which, we have

$$\frac{\pi}{2}F(x) = \int_0^\infty f_c(\lambda)\cos\lambda x\,d\lambda.$$

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$$\Rightarrow \frac{\pi}{2} f(x) = \int_0^1 f_c(\lambda) \cos\lambda x \, d\lambda + \int_1^\infty f_c(\lambda) \cos\lambda x \, d\lambda$$

$$= \int_0^1 (1-\lambda) \cos\lambda x \, d\lambda + \int_1^\infty 0 \cdot \cos\lambda x \, d\lambda.$$

$$= \int_0^1 (1-\lambda) \cos\lambda x \, d\lambda + 0$$

$$= \left[(1-\lambda) \frac{\sin\lambda x}{x} - \int (-1) \frac{\sin\lambda x}{x} \, d\lambda \right]_0^1.$$

$$= \left[(1-\lambda) \frac{\sin\lambda x}{x} + \frac{1}{x} \cdot \left(-\frac{\cos\lambda x}{x} \right) \right]_0^1$$

$$= 0 - \frac{1}{x^2} (\cos x - 1)$$

$$f(x) = \frac{1-\cos x}{x^2}$$

$$f(x) = \frac{2}{\pi x^2} (1-\cos x) \quad \text{Ans}$$

blem (12) Find the Fourier ~~Sine~~ transform of $f(x) = \frac{1}{x}$.

Let $F_S\{f(s)\} = \bar{f}_s(s)$. Then

$$\bar{f}_s(s) = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty \frac{\sin sx}{x} \, dx = \frac{\pi}{2} \quad \text{---} (1)$$

$$\bar{f}_s(s) = \frac{\pi}{2} \quad \left[\text{since } \int_0^\infty \frac{\sin mx}{x} \, dx = \frac{\pi}{2}, \text{ if } m > 0 \right]$$

Ans