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Hilbert Space

PG-Sem III
Paper CG-308

Theorem: Let M be a closed linear subspace of a Hilbert space H , let x be a vector not in M , and let d be the distance from x to M . Then there exists a unique vector y_0 in M such that

$$\|x - y_0\| = d$$

Proof By the definition of distance from x to M , we have $d(x, M) = d = \inf \{ \|x - z\| : z \in M \}$. Then \exists a sequence $\langle y_n \rangle$ of vectors in M s.t. $\|x - y_n\| \rightarrow d$. Suppose two vectors y_m and $y_n \in \langle y_n \rangle$. $\therefore M$ is a linear subspace of H . $\therefore y_m, y_n \in M \Rightarrow \frac{y_m + y_n}{2} \in M$. Hence by definition of d , we have $\|x - \frac{y_m + y_n}{2}\| \geq d$

$$\Rightarrow \|2x - (y_m + y_n)\| \geq 2d \quad \text{--- (1)}$$

Applying the $\| \cdot \|$ law for the vectors $x - y_m$ and $x - y_n$, we have

$$\|(x - y_m) - (x - y_n)\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - \| (x - y_m) + (x - y_n) \|^2$$

$$\Rightarrow \|y_n - y_m\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - \|2x - (y_m + y_n)\|^2 \leq 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4d^2 \quad \text{by (1)}$$

Now $\|x - y_m\| \rightarrow d$ and $\|x - y_n\| \rightarrow d$

$$\therefore 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4d^2 \rightarrow 2d^2 + 2d^2 - 4d^2 = 0$$

Hence as $m, n \rightarrow \infty$, we have $\|y_n - y_m\|^2 \rightarrow 0$

\therefore Cauchy sequence $\langle y_n \rangle$ is a Cauchy sequence in M .

Since H is complete and M is a closed subspace of H

$\therefore M$ is also complete. Hence the Cauchy sequence $\langle y_n \rangle$ in M is also a convergent sequence in M .

$\therefore \exists$ a vector y_0 in M s.t. $y_n \rightarrow y_0$.

$$\text{Now } \|x - y_0\| = \|x - \lim y_n\| = \lim \|x - y_n\| = d.$$

Hence y_0 is a vector in M s.t. $\|x - y_0\| = d$

Uniqueness of y_0 let y_1 and y_2 are two vectors in M s.t. $\|x - y_1\| = d$ and $\|x - y_2\| = d$. Then we have to show $y_1 = y_2$.

$\therefore M$ is a subspace of H

$$\therefore y_1, y_2 \in M \Rightarrow \frac{y_1 + y_2}{2} \in M.$$

Hence by definition of d , we have

$$\|x - \frac{y_1 + y_2}{2}\| \geq d \Rightarrow \|2x - (y_1 + y_2)\| \geq 2d$$

By the triangle law

$$\|(x - y_1) - (x - y_2)\|^2 = 2\|x - y_1\|^2 + 2\|x - y_2\|^2 -$$

$$\| (x - y_1) + (x - y_2) \|^2$$

$$\Rightarrow \|y_2 - y_1\|^2 = 2\|x - y_1\|^2 + 2\|x - y_2\|^2 - \|2x - (y_1 + y_2)\|^2 \\ \leq 2d^2 + 2d^2 - 4d^2 = 0$$

$$\Rightarrow \|y_2 - y_1\|^2 \leq 0. \text{ But } \|y_2 - y_1\|^2 \geq 0$$

Hence it must have

$$\|y_2 - y_1\|^2 = 0 \Rightarrow y_2 - y_1 = 0$$

$$\Rightarrow y_1 = y_2$$

Proved

Orthogonality let x and y be vectors in a Hilbert space H . Then x is said to be orthogonal to y , written as $x \perp y$, if $(x, y) = 0$

Theorem: Pythagorean theorem If x and y are any two orthogonal vectors in a Hilbert space H , then $\|x+y\|^2 + \|x-y\|^2 = \|x\|^2 + \|y\|^2$

Proof Given $x \perp y$ i.e. $(x, y) = 0$. Then we must have $y \perp x$ i.e. $(y, x) = 0$

$$\begin{aligned}\text{Now } \|x+y\|^2 &= (x+y, x+y) \\ &= (x, x) + (x, y) + (y, x) + (y, y) \\ &= \|x\|^2 + 0 + 0 + \|y\|^2 \\ &= \|x\|^2 + \|y\|^2\end{aligned}$$

$$\begin{aligned}\text{Also } \|x-y\|^2 &= (x-y, x-y) \\ &= (x, x) - (x, y) - (y, x) + (y, y) \\ &= \|x\|^2 - 0 - 0 + \|y\|^2 \\ &= \|x\|^2 + \|y\|^2\end{aligned}$$

Hence the result.

Definition: A vector x is said to be orthogonal to a non-empty subset S of a Hilbert space H written as $x \perp S$ if $x \perp y, \forall y \in S$. Two non-empty subsets S_1 and S_2 of a Hilbert space H are said to be orthogonal, written as $S_1 \perp S_2$ if $x \perp y, \forall x \in S_1, \text{ and } y \in S_2$.

• Orthogonal Complement: Let S be a non empty subset of a Hilbert space H . The Orthogonal Complement of S , written as S^\perp and read as S perpendicular, is defined by

$$S^\perp = \{x \in H : x \perp y \ \forall y \in S\}$$

i.e. S^\perp is the set of all vectors in H , which are orthogonal to every vector in S .

Orthogonal complement of an orthogonal complement:

Let S be any non-empty subset of a Hilbert space H . Then S^\perp is a subspace of H . We define $(S^\perp)^\perp$, written as $S^{\perp\perp}$, defined by

$$S^{\perp\perp} = \{x \in H : (x, y) = 0 \ \forall y \in S^\perp\}.$$

Theorem: If S, S_1, S_2 are non empty subsets of a Hilbert space H , then prove that

- (i) $\{0\}^\perp = H$ (ii) $H^\perp = \{0\}$ (iii) $S \cap S^\perp \subset \{0\}$
 (iv) $S_1 \subset S_2 \Rightarrow S_2^\perp \subset S_1^\perp$ (v) $S \subset S^{\perp\perp}$

Proof (i) We show that $H \subset \{0\}^\perp$. Let $x \in H$,

Since $(x, 0) = 0$, so $x \in \{0\}^\perp$.

Thus $x \in H \Rightarrow x \in \{0\}^\perp \therefore H \subset \{0\}^\perp$

But $\{0\}^\perp \subset H$. Hence $\{0\}^\perp = H$.

(ii) Let $x \in H^\perp$. Then by definition of H^\perp ,

$$(x, y) = 0 \ \forall y \in H$$

Taking $y = x$, we have $(x, x) = 0 \Rightarrow \|x\|^2 = 0 \Rightarrow x = 0$

Thus $x \in H^\perp \Rightarrow x = 0$. $\therefore H^\perp = \{0\}$.

(iii) let $x \in S \cap S^\perp \Rightarrow x \in S$ and $x \in S^\perp$.
 Since $x \in S^\perp$, so x is orthogonal to every vector in S . Particularly x is orthogonal to x because $x \in S$.

Now $(x, x) = 0 \Rightarrow \|x\|^2 = 0 \Rightarrow x = 0$. Thus 0 is the only vector which can belong to both S and S^\perp .

$$\therefore S \cap S^\perp \subset \{0\}.$$

(iv) let $S_1 \subset S_2$, we have

$x \in S_2^\perp \Rightarrow x$ is orthogonal to every vector in S_2
 $\Rightarrow x$ is orthogonal to every vector in S_1
 ($\because S_1 \subset S_2$)

$$\Rightarrow x \in S_1^\perp$$

$$\therefore S_2^\perp \subset S_1^\perp.$$

(v) let $x \in S$. Then $(x, y) = 0 \quad \forall y \in S^\perp$.

Then by definition of $(S^\perp)^\perp$, $x \in (S^\perp)^\perp$.

$$\text{Thus } x \in S \Rightarrow x \in S^{\perp\perp}$$

$$\therefore S \subset S^{\perp\perp}. \quad \underline{\text{Q.E.D.}}$$

Theorem Projection Theorem If M is a closed linear subspace of a Hilbert space H ,
 then $H = M \oplus M^\perp$.

Proof: Since M is a subspace of a Hilbert space H , then we know that $M \cap M^\perp = \{0\}$.

Now in order to prove $H = M \oplus M^\perp$, we only check $H = M + M^\perp$.

Since we know that M^\perp is a closed subspace of H .

Also M is given to be closed subspace of H . Then

$M + M^\perp$ is also a closed subspace of H .

Let us put $N = M + M^\perp$. — (1)

From (1) we have $M \subset N$ and $M^\perp \subset N$. Then

we know that $N^\perp \subset M^\perp$ and $N^\perp \subset M^{\perp\perp}$.

$$\Rightarrow N^\perp \subset M^\perp \cap M^{\perp\perp} = \{0\}.$$

$$\begin{aligned} \therefore N^\perp = \{0\} &\Rightarrow (N^\perp)^\perp = \{0\}^\perp \Rightarrow N^{\perp\perp} = H \quad \{\because \{0\}^\perp = H\} \\ &\Rightarrow N = H \quad (\because N = M + M^\perp \text{ is a closed subspace} \\ &\quad \text{of } H \Rightarrow N^{\perp\perp} = N) \end{aligned}$$

Thus $N = M + M^\perp \subseteq H$.

Finally $H = M + M^\perp$ and $M \cap M^\perp = \{0\}$

$$\Rightarrow H = M \oplus M^\perp$$

Theorem: If M is a linear subspace of a Hilbert space H ,

then P.T. M is closed $\Leftrightarrow M = M^{\perp\perp}$.

Proof: Let M be a subspace of a Hilbert space H
and let $M = M^{\perp\perp} = (M^\perp)^\perp$, since we know that

$M \subset M^{\perp\perp}$. Suppose now that this inclusion is proper
i.e. $M \neq M^{\perp\perp}$. Now $M^{\perp\perp}$ is a Hilbert space and M
is a proper closed subspace of $M^{\perp\perp}$. Then
 \exists a non zero vector z_0 in $M^{\perp\perp}$ s.t. $z_0 \perp M$
or $z_0 \in M^\perp$.

$$\text{Now } z_0 \in M^\perp \text{ and } z_0 \in M^{\perp\perp} \Rightarrow z_0 \in M^\perp \cap M^{\perp\perp}. \quad (1)$$

$$\therefore M^\perp \text{ is a subspace of } H. \text{ Then } M^\perp \cap M^{\perp\perp} = \{0\} \quad (2)$$

By (1) & (2) we conclude that $z=0$. It contradicts
that z_0 is a non zero vector. Therefore by inclusion
 $M \subset M^{\perp\perp}$ cannot be proper and we must have $M = M^{\perp\perp}$.

Theorem Let S be a nonempty subset of a Hilbert space H . Then S^\perp is a closed linear subspace of H .

Proof: By definition, $S^\perp = \{x \in H : (x, y) = 0 \ \forall y \in S\}$

$\therefore (0, y) = 0 \ \forall y \in S$, then at least $0 \in S^\perp$ and thus

S^\perp is not empty. Now let $x_1, x_2 \in S^\perp$ and α, β are any two scalars.

Then $(x_1, y) = 0 \ \forall y \in S$ and $(x_2, y) = 0 \ \forall y \in S$.

Now for every $y \in S$, we have

$$(\alpha x_1 + \beta x_2, y) = \alpha(x_1, y) + \beta(x_2, y) = \alpha \cdot 0 + \beta \cdot 0 = 0.$$

$\therefore \alpha x_1 + \beta x_2 \in S^\perp$. Hence S^\perp is a subspace of H .

Now we shall show that S^\perp is a closed subset of H . Now we shall show that if x is any limit of S^\perp , then $x \in S^\perp$ i.e. $(x, y) = 0 \ \forall y \in S$.

Since x is a limit point of S^\perp , then there exists a sequence $\langle x_n \rangle$ of points of S^\perp s.t. $x_n \rightarrow x$.

Now let $y \in S$. Since $x_n \in S^\perp \ \forall n$

$\therefore (x_n, y) = 0, \forall n$, consequently, we have

$$\lim_{n \rightarrow \infty} (x_n, y) = 0 \Rightarrow (\lim_{n \rightarrow \infty} x_n, y) = 0$$

(\because inner product is a continuous mapping)

$$\Rightarrow (x, y) = 0 \quad (\because \lim_{n \rightarrow \infty} x_n = x)$$

Thus if x is a limit point of S^\perp , then $(x, y) = 0, \forall y \in S$.

$\therefore x \in S^\perp$. Hence S^\perp is a closed subset of H .

Proved

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