

FOURIER TRANSFORM (7)

Problem 13 Find sine and cosine transform of $x^n e^{-ax}$

Ans Here $f(x) = x^n e^{-ax}$

$$F_c\{f(x)\} = f_c(s) = \int_0^{\infty} x^n e^{-ax} \cos(sx) dx \quad (1)$$

$$\text{and } F_s\{f(x)\} = f_s(s) = \int_0^{\infty} x^n e^{-ax} \sin(sx) dx \quad (2)$$

We know that $\int_0^{\infty} e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$

$$\therefore \int_0^{\infty} e^{-isx} (x^n e^{-ax}) dx = \int_0^{\infty} e^{-x(a+is)} x^{(n+1)-1} dx = \frac{\Gamma(n+1)}{(a+is)^{n+1}}$$

Taking $a+is = re^{i\theta}$, then we get

$$\int_0^{\infty} e^{-isx} (x^n e^{-ax}) dx = \frac{\Gamma(n+1)}{(re^{i\theta})^{n+1}} = \frac{\Gamma(n+1)}{r^{n+1}} e^{-i(n+1)\theta}$$

$$\Rightarrow \int_0^{\infty} (x^n e^{-ax}) (\cos sx - i \sin sx) dx = \frac{\Gamma(n+1)}{(a^2 + s^2)^{n+1/2}} \begin{Bmatrix} \cos(n+1)\theta \\ -i \sin(n+1)\theta \end{Bmatrix}$$

Equating real and imaginary parts.

$$\text{R.P.} \Rightarrow \int_0^{\infty} (x^n e^{-ax}) \cos(sx) dx = \frac{\Gamma(n+1) \cos(n+1)\theta}{(a^2 + s^2)^{n+1/2}}$$

$$\text{I.P.} \Rightarrow \int_0^{\infty} (x^n e^{-ax}) \sin(sx) dx = \frac{\Gamma(n+1) \sin(n+1)\theta}{(a^2 + s^2)^{n+1/2}}$$

$$\therefore f_c(s) = \frac{\ln \cos(n+1)\theta}{(a^2 + s^2)^{n+1/2}}$$

and $f_s(s) = \frac{\ln \sin(n+1)\theta}{(a^2 + s^2)^{n+1/2}}$ } Ans

Problem (14) Find Fourier sine and cosine transform of $2e^{-5x} + 5e^{-2x}$.

Ans (i) $F_c \{ 2e^{-5x} + 5e^{-2x} \} = 2F_c \{ e^{-5x} \} + 5F_c \{ e^{-2x} \}$

$$= 2 \int_0^{\infty} e^{-5x} \cos sx \, dx + 5 \int_0^{\infty} e^{-2x} \cos sx \, dx$$

$$= 2 \cdot \frac{5}{s^2 + 25} + 5 \cdot \frac{2}{s^2 + 4} \left(\because \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \right)$$

$$= 10 \left(\frac{1}{s^2 + 25} + \frac{1}{s^2 + 4} \right) \underline{\text{Ans}}$$

(ii) $f_s \{ 2e^{-5x} + 5e^{-2x} \} = 2f_s \{ e^{-5x} \} + 5f_s \{ e^{-2x} \}$

$$= 2 \int_0^{\infty} e^{-5x} \sin sx \, dx + 5 \int_0^{\infty} e^{-2x} \sin sx \, dx$$

$$= 2 \cdot \frac{s}{s^2 + 25} + 5 \cdot \frac{s}{s^2 + 4} \left(\because \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \right)$$

$$= s \left(\frac{2}{s^2 + 25} + \frac{5}{s^2 + 4} \right)$$

\therefore Sine transform is $s \left(\frac{2}{s^2 + 25} + \frac{5}{s^2 + 4} \right)$

and cosine transform is $10 \left(\frac{1}{s^2 + 25} + \frac{1}{s^2 + 4} \right) \underline{\text{Ans}}$

Question (15) find $f(x)$ if its Fourier sine transform is $\frac{s}{1+s^2}$
 If $\bar{f}_s(s) = \frac{s}{1+s^2} = F_s\{f(x)\}$, then find $f(x)$.

OR, Find $F_s^{-1}\left\{\frac{s}{1+s^2}\right\}$.

Answer $\therefore f(x) = F_s^{-1}\{\bar{f}_s(s)\} = F_s^{-1}\left\{\frac{s}{1+s^2}\right\}$

$$\therefore f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{s}{1+s^2} \cdot \sin sx \cdot ds$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{s^2}{s(1+s^2)} \cdot \sin sx \, ds = \frac{2}{\pi} \int_0^{\infty} \frac{(s^2+1)-1}{s(1+s^2)} \cdot \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{(s^2+1)-1}{s(s^2+1)} \cdot \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\sin sx}{s} \, ds - \frac{2}{\pi} \int_0^{\infty} \frac{\sin sx}{s(s^2+1)} \, ds$$

$$= \frac{2}{\pi} \cdot \frac{1}{2} - \frac{2}{\pi} \int_0^{\infty} \frac{\sin sx}{s(1+s^2)} \, ds \quad \text{--- (1)}$$

$$\Rightarrow \frac{df(x)}{dx} = 0 - \frac{2}{\pi} \int_0^{\infty} \frac{s \cdot \cos sx}{s(1+s^2)} \, ds$$

$$\Rightarrow \frac{df}{dx} = -\frac{2}{\pi} \int_0^{\infty} \frac{\cos sx}{1+s^2} \, ds \quad \text{--- (2)}$$

$$\Rightarrow \frac{d^2 f}{dx^2} = -\frac{2}{\pi} \int_0^{\infty} -s \cdot \frac{\sin sx}{1+s^2} \, ds = \frac{2}{\pi} \int_0^{\infty} \frac{s}{1+s^2} \sin sx \, ds = f$$

$$\Rightarrow \frac{d^2 f}{dx^2} - f = 0 \Rightarrow (D^2 - 1)f = 0$$

$$A.E. \text{ is } m^2 - 1 = 0$$

$$\Rightarrow m = \pm 1$$

\therefore Complementary solution is

$$f = A \cdot e^x + B e^{-x} \quad \text{--- (3)}$$

$$\Rightarrow \frac{df}{dx} = A e^x - B e^{-x} \quad \text{--- (4)}$$

when $x=0$ then $f=1$ [by (1)].

From these conditions, (3) is

$$A + B = 1 \quad \text{--- (5)}$$

Now by (2), when $x=0$

$$\frac{df}{dx} = -\frac{2}{\pi} \left[\tan^{-1} s \right]_{s=0}^{\infty} = -1$$

$$\therefore \frac{df}{dx} = -1, \text{ when } x=0$$

Now by (4), applying these conditions

$$A - B = -1 \quad \text{--- (6)}$$

Solving (5) & (6), we have

$$A = 0, B = 1.$$

Now by (3), we have

$$f = e^{-x} \quad \underline{\underline{\text{Ans}}}$$