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PG - Sem III
Paper CC-308

Hilbert Space

Theorem: Let M be a closed linear subspace of a Hilbert space H, let x be a vector not in M, and let d be the distance from x to M. Then there exists a unique vector y_0 in M such that

$$\|x - y_0\| = d$$

Proof By the definition of distance from x to M, we have $d(x, M) = d = \inf\{\|x-z\| : z \in M\}$

Then $\exists$ a sequence $\langle y_n \rangle$ of vectors in M s.t. $\|x - y_n\| \to d$. Suppose two vectors y_m and $y_n \in \langle y_n \rangle$.

$\because$ M is a linear subspace of H.

$\therefore$ $y_m, y_n \in M \Rightarrow \frac{y_m + y_n}{2} \in M$. Hence by definition of d, we have $\left\| x - \frac{y_m + y_n}{2} \right\| \geq d$

$$\Rightarrow \|2x - (y_m + y_n)\| \geq 2d \longrightarrow (1)$$

Applying the $\|^{gm}$ law for the vectors $x - y_m$ and $x - y_n$, we have

$$\|(x - y_n) - (x - y_m)\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - \|(x - y_m) + (x - y_n)\|^2$$

$$\Rightarrow \|y_n - y_m\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - \|2x - (y_m + y_n)\|^2$$

$$\leq 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4d^2 \quad \text{by (1)}$$

Now $\|x - y_m\| \to d$ and $\|x - y_n\| \to d$

$\therefore 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0$

Hence as $m, n \to \infty$, we have $\|y_n - y_m\|^2 \to 0$

$\therefore$ Cauchy sequence $\langle y_n \rangle$ is a Cauchy sequence in M.
Since H is complete and M is a closed subspace of H
$\therefore$ M is also complete. Hence the Cauchy sequence $\langle y_n \rangle$ in M is also a convergent sequence in M.
$\therefore$ $\exists$ a vector y_0 in M s.t. $y_n \to y_0$.

Now $\|x - y_0\| = \|x - \lim y_n\| = \lim \|x - y_n\| = d$.
Hence y_0 is a vector in M s.t. $\|x - y_0\| = d$

<u>Uniqueness of y_0</u> Let y_1 and y_2 are two vectors in M s.t. $\|x - y_1\| = d$ and $\|x - y_2\| = d$. Then we have to show $y_1 = y_2$.

$\because$ M is a subspace of H

$\therefore$ $y_1, y_2 \in M \Rightarrow \dfrac{y_1 + y_2}{2} \in M$.

Hence by definition of d, we have

$$\left\| x - \frac{y_1 + y_2}{2} \right\| \geq d \Rightarrow \| 2x - (y_1 + y_2) \| \geq 2d$$

By the $\|^{gm}$ law

$$\|(x - y_1) - (x - y_2)\|^2 = 2\|x - y_1\|^2 + 2\|x - y_2\|^2 - \|(x - y_1) + (x - y_2)\|^2$$

$$\Rightarrow \|y_2 - y_1\|^2 = 2\|x - y_1\|^2 + 2\|x - y_2\|^2 - \|2x - (y_1 + y_2)\|^2$$
$$\leq 2d^2 + 2d^2 - 4d^2 = 0$$

$\Rightarrow \|y_2 - y_1\|^2 \leq 0$. But $\|y_2 - y_1\|^2 \geq 0$

Hence it must have

$$\|y_2 - y_1\|^2 = 0 \Rightarrow y_2 - y_1 = 0$$
$$\Rightarrow y_1 = y_2$$

Proved

Orthogonality Let x and y be vectors in a Hilbert space H. Then x is said to be orthogonal to y, written as $x \perp y$, if $(x, y) = 0$

Theorem: Pythagorean theorem If x and y are any two orthogonal vectors in a Hilbert space H, then $\|x+y\|^2 \doteq \|x-y\|^2 = \|x\|^2 + \|y\|^2$

Proof Given $x \perp y$ ie $(x, y) = 0$. Then we must have $y \perp x$ ie $(y, x) = 0$

$$\begin{aligned}
\text{Now } \|x+y\|^2 &= (x+y, x+y) \\
&= (x, x) + (x, y) + (y, x) + (y, y) \\
&= \|x\|^2 + 0 + 0 + \|y\|^2 \\
&= \|x\|^2 + \|y\|^2
\end{aligned}$$

$$\begin{aligned}
\text{Also } \|x-y\|^2 &= (x-y, x-y) \\
&= (x, x) - (x, y) - (y, x) + (y, y) \\
&= \|x\|^2 - 0 - 0 + \|y\|^2 \\
&= \|x\|^2 + \|y\|^2
\end{aligned}$$

Hence the result.

Definition: A vector x is said to be orthogonal to a non-empty subset S of a Hilbert space H written as $x \perp S$ if $x \perp y$, $\forall y \in S$.

Two non-empty subsets S_1 and S_2 of a Hilbert space H are said to be orthogonal, written as $S_1 \perp S_2$ if $x \perp y$ $\forall x \in S_1$ and $y \in S_2$.

Orthogonal Complement: Let S be a non empty subset of a Hilbert space H. The orthogonal complement of S, written as $S^{\perp}$ and read as S perpendicular, is defined by

$$S^{\perp} = \{x \in H : x \perp y \ \forall\, y \in S\}$$

i.e. $S^{\perp}$ is the set of all vectors in H, which are orthogonal to every vector in S.

Orthogonal complement of an orthogonal complement:

Let S be any non-empty subset of a Hilbert space H. Then $S^{\perp}$ is a subspace of H. We define $(S^{\perp})^{\perp}$, written as $S^{\perp\perp}$, defined by

$$S^{\perp\perp} = \{x \in H : (x, y) = 0 \ \forall\, y \in S^{\perp}\}.$$

Theorem: If S, S_1, S_2 are non empty subsets of a Hilbert space H, then prove that

(i) $\{0\}^{\perp} = H$ (ii) $H^{\perp} = \{0\}$ (iii) $S \cap S^{\perp} \subset \{0\}$

(iv) $S_1 \subset S_2 \Rightarrow S_2^{\perp} \subset S_1^{\perp}$ (v) $S \subset S^{\perp\perp}$

Proof (i) We show that $H \subset \{0\}^{\perp}$. Let $x \in H$.

Since $(x, 0) = 0$, so $x \in \{0\}^{\perp}$.

Thus $x \in H \Rightarrow x \in \{0\}^{\perp}$ $\therefore H \subset \{0\}^{\perp}$

But $\{0\}^{\perp} \subset H$. Hence $\{0\}^{\perp} = H$.

(ii) Let $x \in H^{\perp}$. Then by definition of $H^{\perp}$,

$(x, y) = 0 \ \forall\, y \in H$

Taking $y = x$, we have $(x, x) = 0 \Rightarrow \|x\|^2 = 0 \Rightarrow x = 0$

Thus $x \in H^{\perp} \Rightarrow x = 0$. $\therefore H^{\perp} = \{0\}$.

(iii) Let $x \in S \cap S^\perp \Rightarrow x \in S$ and $x \in S'$
Since $x \in S'$, so x is orthogonal to every vector in S. Particularly x is orthogonal to x because $x \in S$.
Now $(x, x) = 0 \Rightarrow ||x||^2 = 0 \Rightarrow x = 0$. Thus 0 is the only vector which can belong to both S and $S^\perp$.

$$\therefore \quad S \cap S^\perp \subset \{0\}.$$

(iv) Let $S_1 \subset S_2$, we have

$x \in S_2^\perp \Rightarrow x$ is orthogonal to every vector in S_2

$\Rightarrow x$ is orthogonal to every vector in S_1 $(\because S_1 \subset S_2)$

$\Rightarrow x \in S_1^\perp$

$$\therefore \quad S_2^\perp \subset S_1^\perp.$$

(v) Let $x \in S$. Then $(x, y) = 0 \ \forall \ y \in S^\perp$.
Then by definition of $(S^\perp)^\perp$, $x \in (S^\perp)^\perp$.
Thus $x \in S \Rightarrow x \in S^{\perp\perp}$

$$\therefore \quad S \subset S^{\perp\perp}.$$

<u>Sn</u>

<u>Theorem</u> <u>Projection Theorem</u> If M is a closed linear subspace of a Hilbert space H,
Then $H = M \oplus M^\perp$.

<u>Proof</u>: Since M is a subspace of a Hilbert space H, then we know that $M \cap M^\perp = \{0\}$.
Now In order to prove $H = M \oplus M^\perp$, we only check $H = M + M^\perp$.

Since we know that $M^\perp$ is a closed subspace of H.
Also M is given to be closed subspace of H. Then
$M + M^\perp$ is also a closed subspace of H.
Let us put $N = M + M^\perp$. —— (1)
From (1) we have $M \subset N$ and $M^\perp \subset N$. Then
we know that $N^\perp \subset M^\perp$ and $N^\perp \subset M^{\perp\perp}$.

$$\Rightarrow N^\perp \subset M^\perp \cap M^{\perp\perp} = \{0\}.$$

$$\therefore N^\perp = \{0\} \Rightarrow (N^\perp)^\perp = \{0\}^\perp \Rightarrow N^{\perp\perp} = H \quad \{\because \{0\}^\perp = H\}$$

$$\Rightarrow N = H \quad (\because N = M + M^\perp \text{ is a closed subspace of } H \Rightarrow N^{\perp\perp} = N)$$

Thus $N = M + M^\perp = H$.
Finally $H = M + M^\perp$ and $M \cap M^\perp = \{0\}$

$$\Rightarrow H = M \oplus M^\perp$$

Theorem: If M is a linear subspace of a Hilbert space H. Then P.T. M is closed $\Leftrightarrow M = M^{\perp\perp}$.

Proof: Let M be a subspace of a Hilbert space H and let $M = M^{\perp\perp} = (M^\perp)^\perp$, since we know that $M \subset M^{\perp\perp}$. Suppose now that this inclusion is proper i.e. $M \neq M^{\perp\perp}$. Now $M^{\perp\perp}$ is a Hilbert space and M is a proper closed subspace of $M^{\perp\perp}$. Then $\exists$ a non zero vector z_0 in $M^{\perp\perp}$ s.t. $z_0 \perp M$ or $z_0 \in M^\perp$.
Now $z_0 \in M^\perp$ and $z_0 \in M^{\perp\perp} \Rightarrow z_0 \in M^\perp \cap M^{\perp\perp}$. —— (1)
$\because M^\perp$ is a subspace of H. Then $M^\perp \cap M^{\perp\perp} = \{0\}$ —— (2)
By (1) & (2) we conclude that $z = 0$. It contradicts that z_0 is a non zero vector. Therefore by inclusion $M \subset M^{\perp\perp}$ cannot be proper and we must have $M = M^{\perp\perp}$.

Theorem: Let S be a non empty subset of a Hilbert space H. Then $S^{\perp}$ is a closed linear subspace of H.

Proof: By definition, $S^{\perp} = \{x \in H : (x, y) = 0 \ \forall y \in S\}$

$\because (0, y) = 0 \ \forall y \in S$, Then atleast $0 \in S^{\perp}$ and thus $S^{\perp}$ is not empty. Now let $x_1, x_2 \in S^{\perp}$ and α, β are any two scalars.

Then $(x_1, y) = 0 \ \forall y \in S$ and $(x_2, y) = 0 \ \forall y \in S$

Now for every $y \in S$, we have

$(\alpha x_1 + \beta x_2, y) = \alpha(x_1, y) + \beta(x_2, y) = \alpha 0 + \beta 0 = 0.$

$\therefore \alpha x_1 + \beta x_2 \in S^{\perp}$. Hence $S^{\perp}$ is a subspace of H.

Now we shall show that $S^{\perp}$ is a closed subset of H. Now we shall show that if x is any limit of $S^{\perp}$, then $x \in S^{\perp}$ i.e. $(x, y) = 0 \ \forall y \in S$.

Since x is a limit point of $S^{\perp}$, Then there exists a sequence $\langle x_n \rangle$ of points of $S^{\perp}$ s.t. $x_n \to x$.

Now let $y \in S$. Since $x_n \in S^{\perp} \ \forall n$

$\therefore (x_n, y) = 0, \forall n$, Consequently, we have

$\lim (x_n, y) = 0 \Rightarrow (\lim x_n, y) = 0$

($\because$ Inner product is a continuous mapping)

$\Rightarrow (x, y) = 0 \quad (\because \lim x_n = x)$

Thus if x is a limit point of $S^{\perp}$, Then $(x, y) = 0 \ \forall y \in S$.

$\therefore x \in S^{\perp}$. Hence $S^{\perp}$ is a closed subset of H.

Proved

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