Field Adjunction: Suppose K is an extension field of a field F. Let $a \in K$. Suppose C is collection of all subfields of K containing both F and a.

Obviously C is non-empty because at least K itself is in C. Let F(a) denotes intersection of of all those subfield of K which are in C.

Then F(a) is also a subfield of K and F(a) contains both F and a. So F(a) is also a member of C. Also F(a) is smallest field containing both F and a. We call F(a) is a subfield obtained by adjoining a to F.

Construction of F(a): Suppose K is extension of a field F. Let $a \in K$, and let

$$U = \left\{ \frac{k_0 a^n + k_1 a^{n-1} + \cdots + k_n}{l_0 a^m + l_1 a^{m-1} + \cdots + l_m} \;\middle|\; \begin{array}{l} \text{where } k_i\text{'s } \& \ l_j \in F \text{ and} \\ l_0 a^m + l_1 a^{m-1} + \cdots + l_m \neq 0 \text{ of } K \\ n \,\&\, m \text{ are non-negative integers} \end{array} \right\}$$

Obviously U is a subfield of K.

[Note: It can easily be proved that U is the smallest subfield of K containing F & a ie $U = F(a)$]

Simple field Extension: The extension K of a field F is called simple extension of F if $K = F(a)$ for some $a \in K$.

Let K be an extension of field F. Let $a \,\&\, b \in K$. Let $T = F(a)$. Since F(a) is subfield of K, therefore K is also an extension of F(a). Now W be subfield of K by adjoining b to F(a), then $W = (F(a))(b)$.

We denote this $(F(a))(b)$ by $F(a,b)$.

Similarly we define $F(b,a)$.

Obviously $F(a,b) = F(b,a)$.

So we can simply say that $F(a,b)$ is smallest subfield of K obtained by adjoining a & b to F.

Similarly, if $a_1, a_2, \dots a_n \in K$, then $F(a_1, a_2, \dots a_n)$ is smallest subfield of K obtained by adjoining $a_1, a_2, \dots a_n$ to F.

Algebraic Field Extension:

Let $q(x) \in F[x]$ the ring of polynomials in x over the field F. Let $q(x) = \alpha_0 x^m + \alpha_1 x^{m-1} + \dots + \alpha_m$. Suppose K is an extension of F and $b \in K$, then by $q(b)$ we mean the element $\alpha_0 b^m + \alpha_1 b^{m-1} + \dots + \alpha_m$ of K. $q(b)$ is also called value of $q(x)$ at b.

If $q(b) = 0$ then b is called zero of $q(x)$.

Algebraic element: Let K be an extension of a field F. An element $a \in K$ is said to be algebraic over F if there is a non-zero polynomial $p(x) \in F[x]$ for which $p(a) = 0$.

In other words $a \in K$ is algebraic over F if $\exists$ elements $\beta_0, \beta_1, \dots \beta_n \in F$ such that

$$\beta_0 a^n + \beta_1 a^{n-1} + \dots + \beta_n = 0.$$

Transcendental element: Let K is an extension field of a field F. An element $a \in K$ is said to be transcendental if it is not algebraic over F.

Defn: A complex number is said to be algebraic number if it is algebraic over field of rational numbers.

Minimal polynomial of an algebraic element:

Let K be an extension of a field F. Let $a \in K$ be algebraic over F. Suppose $p(x)$ is a polynomial over F of lowest positive degree satisfied by a. Then $p(x)$ is called a minimal polynomial for a over F.

Let us impose the restriction on minimal polynomial for 'a' over F that it should be monic. i.e., in it the coefficient of highest power of x is 1. Then we can say that minimal polynomial for a over F is unique.

Theorem: Let $a \in K$ be algebraic over F. Then any two minimal polynomials for 'a' over F are equal.

Proof: Let $x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n$ & $x^n + \beta_1 x + \cdots + \beta_n$ be two minimal polynomials of 'a' over F.

Then

$$a^n + \alpha_1 a^{n-1} + \cdots + \alpha_n = 0 \quad \text{——— (1)}$$

$$\& \quad a^n + \beta_1 a^{n-1} + \cdots + \beta_n = 0 \quad \text{——— (2)}$$

From 1 & 2

$$a^n + \alpha_1 a^{n-1} + \cdots + \alpha_n = a^n + \beta_1 a^{n-1} + \cdots + \beta_n$$

$$\Rightarrow (\alpha_1 - \beta_1)\cdot{}^{n-1} + (\alpha_2 - \beta_2)\,{}^{n-2} + \cdots + (\alpha_n - \beta_n) = 0$$

$$\Rightarrow \gamma_1 a^{n-1} + \gamma_2 a^{n-2} + \cdots + \gamma_n = 0$$

where $\gamma_1 = \alpha_1 - \beta_1$, $\gamma_2 = \alpha_2 - \beta_2 \cdots$ $\gamma_n = \alpha_n - \beta_n \in F$

$\Rightarrow$ a satisfies a polynomial $q(x) = \gamma_1 x^{n-1} + \gamma_2 x^{n-2} + \cdots + \gamma_n$ of $F[x]$ whose degree is less than n.

$\Rightarrow$ $q(x)$ must be zero polynomial because the degree of minimal polynomial for 'a' over F is n.

$$\Rightarrow \gamma_1 = 0, \quad \gamma_2 = 0, \quad \cdots \quad \gamma_n = 0$$

$$\Rightarrow \alpha_1 - \beta_1 = 0, \quad \alpha_2 - \beta_2 = 0, \quad \cdots \quad \alpha_n - \beta_n = 0$$

$$\Rightarrow \alpha_1 = \beta_1, \quad \alpha_2 = \beta_2 \quad \cdots \quad \alpha_n = \beta_n$$

Hence the two polynomials $x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n$ and $x^n + \beta_1 x^{n-1} + \cdots + \beta_n$ are same polynomials

Proved