Normalised Function:

<Q> Define normalised function. [R.U. 73, 79]

If $\alpha(x)$ be a function of bounded variation in $a \le x \le b$. It is said to be normalised, if

$$\alpha(a) = 0$$

and $$\alpha(x) = \frac{\alpha(x+) + \alpha(x-)}{2}, \quad a < x < b$$

Thus, a function $\alpha(x)$, define in $[a, b]$ to be normalised, if the following conditions are satisfied:-

(i) $\alpha(x)$ is of b.v. in $[a, b]$

(ii) $\alpha(x) = 0$ for $x = a$ ie $\alpha(a) = 0$

(iii) $\alpha(x) = \dfrac{\alpha(x+) + \alpha(x-)}{2}$, for $a < x < b$.

Theorem 8(a):- P.T. normalisation of a function $\alpha(x)$ in $[a, b]$ does not change the value of the integral of a continuous funct. $f(x)$ in $[a, b]$ wrt. $\alpha(x)$

[R.U. 79, 73, 85]

Equivalently, If $f(x)$ be continuous and $\alpha(x)$ is of b.v. in $a \le x \le b$, then $\exists$ a normalised funct. $\alpha^*(x)$ of b.v. in $a \le x \le b$ such that

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\, d\alpha^*(x).$$

: We define a function α^* in $[a, b]$ as below

$$\alpha^*(a) = 0$$

$$\alpha^*(x) = \frac{\alpha(x+) + \alpha(x-)}{2} - \alpha(a), \quad a < x <$$

$$\alpha^*(b) = \alpha(b) - \alpha(a)$$

First, we shall prove that α^* is a normalised function. for which we have to prove

(i) $\alpha^*(a) = 0$

(ii) $\alpha^*(x) = \dfrac{\alpha^*(x+) + \alpha^*(x-)}{2}$; $a < x < b$

(iii) $\alpha^*(x)$ is of b.v. in $[a,b]$

Now by definition of α^* itself $\Rightarrow$ the required condition (i) for normalisation of α^*

As $\alpha(x)$ is of b.v. in $[a,b]$, $\alpha^*(x)$ will also be of b.v. in $[a,b]$, by definition of $\alpha^*(x)$

Thus we have to prove only condition (ii) for normalisation of α^*

Proof of (ii)

$$\alpha^*(x+) = \lim_{h\to 0} \alpha^*(x+h)$$

(* For justification see the note at end of Ans.)

$$= \lim_{h\to 0} \frac{\alpha[(x+h)+] + \alpha[(x+h)-]}{2}$$

$$= \frac{\alpha(x+) + \alpha(x+)}{2} - \alpha(a)$$

$$= \alpha(x+) - \alpha(a)$$

Similarly: $\alpha^*(x-) = \lim_{h\to 0} \alpha^*(x-h)$

$$\Rightarrow \alpha^*(x-) = \alpha(x-) - \alpha(a)$$

$$\therefore \alpha^*(x+) + \alpha^*(x-) = \alpha(x+) + \alpha(x-) - 2\alpha(a)$$

$$\Rightarrow \frac{\alpha^*(x+) + \alpha^*(x-)}{2} = \frac{\alpha(x+) + \alpha(x-)}{2} - \alpha(a)$$

$$= \alpha^*(x) \qquad a < x < b$$

Hence the function α^* defined on $[a,b]$ is normalised.

Thus corresponding to a function $\alpha(x)$ of b.v. in $[a,b]$, there exists a normalised function $\alpha^*(x)$ in $[a,b]$

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Now we are left to prove that
$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\,d\alpha^*(x)$$

We define a function α_1 in $[a,b]$ as below :-
$\alpha_1(x) = \alpha(x) - \alpha(a) - \alpha^*(a)$
$\therefore \alpha_1(a) = \alpha(a) - \alpha(a) - \alpha^*(a) = 0$
and $\alpha_1(b) = \alpha(b) - \alpha(a) - \alpha^*(b) = 0$
$\{\because \alpha^*(b) = \alpha(b) - \alpha(a)$, by definition$\}$

As $\alpha(x), \alpha^*(x)$ both are of b.v. in $[a,b]$
and $\alpha_1(x)$ is also of b.v. in $[a,b]$
Hence $\alpha_1(x)$ has infinitely many point of continuity in $[a,b]$ & these points of continuity are also points of continuity of $\alpha(x)$ and $\alpha^*(x)$.

$\therefore x \in]a,b[$, x is a point of continuity of $\alpha(x)$ & $\alpha^*(x)$
$\Rightarrow \alpha(x+) = \alpha(x-) = \alpha(x)$
and $\alpha^*(x+) = \alpha^*(x-) = \alpha^*(x)$
$$\therefore \alpha_1(x) = \alpha(x) - \alpha(a) - \alpha^*(x)$$
$$= \alpha(x) - \alpha(a) - \left[\frac{\alpha(x+) + \alpha(x-)}{2} - \alpha(a)\right]$$
$$= \alpha(x) - \alpha(a) - \left[\frac{\alpha(x) + \alpha(x)}{2} - \alpha(a)\right]$$
(by above)
$$= \alpha(x) - \alpha(a) - \alpha(x) + \alpha(a)$$
$$= 0$$
$\Rightarrow \alpha_1(x) = 0$, $\forall x \in E = \{a = x_0, x_1, x_2, \dots, x_n = b\}$, each x_i is a point of continuity of $\alpha_1(x)$
i.e. $\alpha_1(x)$ has constant value zero in E

But E being infinite and bounded has limit points in $[a,b]$, so E is dense in $[a,b]$

Thus, we have $f(x)$ is continuous value zero at a set of points and which is dense in (a,b)

$$\Rightarrow \int_a^b f(x)\, d\alpha_1(x) = 0$$

$$\Rightarrow \int_a^b f(x)\, d\left[\alpha(x) - \alpha(a) - \alpha^*(x)\right] = 0$$

$$\Rightarrow \int_a^b f(x)\, d\alpha(x) - \int_a^b f(x)\, d\alpha^*(x) = 0 \quad \{\because \alpha(a) = 0\}$$

$$\Rightarrow \int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\, d\alpha^*(x)$$

Hence the theorem

Note :- $\lim_{h\to 0} \alpha(x+h)+ = \lim_{h\to 0} \lim_{h_1\to 0} [\alpha(x+h)+h_1]$

$$= \lim_{h, h_1 \to 0} \alpha(x+h+h_1) = \alpha(x+)$$

Similarly $\lim_{h\to 0} \alpha(x+h) - = \lim_{h, h_1\to 0} \alpha\{x+(h-h_1)\}$

$$= \alpha(x+)$$

<u>Theorem 8(b)</u> If $f(x)$ is continuous, $\alpha(x)$ is a normalised function of b.v. in $a \le x \le b$ then the function $F(x) = \int_a^x f(t)\, d\alpha(t)$ is also normalised.

[R.U. 70, 83]

Proof :- $F(x) = \int_a^x$ will be normalised in $[a, b]$ iff

(1) $F(a) = 0$

(2) $F(x)$ is of b.v. in $[a, b]$

(3) $F(x) = \dfrac{F(x+) + F(x-)}{2}$, for $a < x < b$.

~~Pro~~ For (1)

$$\because F(x) = \int_a^x f(t)\, d\alpha(t)$$

$$\therefore F(a) = \int_a^a f(t)\, d\alpha(t) = 0$$