Problem (4) Using Parseval's identity, show that (i) $\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$ (ii) $\int_0^\infty \frac{x^2}{(x^2+1)^2}dx = \frac{\pi}{4}$

Ans (i) let $F(x) = e^{-x}$, then

$$F_c(F) = f_c(s) = \int_0^\infty F(x)\cdot \cos(sx)\,dx$$

$$= \int_0^\infty e^{-x}\cos(sx)\,dx = \frac{1}{1+s^2}$$

Now by Parseval's identity

$$\frac{2}{\pi}\int_0^\infty |f_c(s)|^2 ds = \int_0^\infty |F(x)|^2 dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \left(\frac{1}{1+s^2}\right)^2 ds = \int_0^\infty (e^{-x})^2 dx = \int_0^\infty e^{-2x}dx$$

$$= \left[\frac{e^{-2x}}{-2}\right]_0^\infty = \frac{1}{2}$$

$$\Rightarrow \int_0^\infty \left(\frac{1}{1+s^2}\right)^2 ds = \frac{\pi}{4}$$

(ii) let $F(x) = \frac{x}{1+x^2}$, then

$$F_s\{F(x)\} = f_s(s) = \int_0^\infty F(x)\cdot \sin(sx)\,dx$$

$$f_s(s) = \int_0^\infty \left(\frac{x}{1+x^2}\right)\sin sx\,dx.$$

For finding this, firstly we shall determine cosine transform of $\frac{1}{1+x^2}$.

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$$F_c\left\{\frac{1}{1+x^2}\right\} = \int_0^\infty \frac{\cos sx}{1+x^2}dx = I \text{ (say)}$$

$$I = \int_0^\infty \frac{\cos sx}{1+x^2}dx \quad \text{——— (1)}$$

Differentiating w.r.t. s,

$$\frac{dI}{ds} = \int_0^\infty \frac{-x\sin sx}{1+x^2}\,dx = F_s\left\{\frac{-x}{1+x^2}\right\} \quad \text{--- (2)}$$

$$= -\int_0^\infty \frac{x}{1+x^2}\sin(sx)\,dx$$

$$= -\int_0^\infty \frac{x^2}{x(1+x^2)}\sin(sx)\,dx = -\int_0^\infty \frac{(1+x^2-1)\sin(sx)}{x(1+x^2)}\,dx$$

$$= -\int_0^\infty \frac{\sin(sx)}{x}\,dx + \int_0^\infty \frac{\sin(sx)}{x(1+x^2)}\,dx$$

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin(sx)}{x(1+x^2)}\,dx \quad \text{--- (3)}$$

Again diffing w.r.t. s

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos(sx)}{x(1+x^2)}\,dx = I$$

$$\frac{d^2I}{ds^2} - I = 0$$

Its Complementry Solution is

$$I = Ae^{-s} + Be^{s} \quad \text{--- (4)}$$

Putting $s=0$ in (1), $I = \int_0^\infty \frac{1}{1+x^2}\,dx = \left(\tan^{-1}x\right)_0^\infty = \frac{\pi}{2}$

$\therefore I = \frac{\pi}{2}$, when $s=0$

Now by (4), applying this property

$$\frac{\pi}{2} = A+B \quad \text{--- (5)}$$

Putting $s=0$ in (3), $\frac{dI}{ds} = -\frac{\pi}{2}+0$

Now by (4), $\frac{dI}{ds} = -Ae^{-s} + Be^{s}$

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Putting $s=0$, we have

$$-\frac{\pi}{2} = -A+B \quad \text{—— (6)}$$

Solving (5) & (6), we have

$B=0$, $A=\frac{\pi}{2}$, Now by (4)

$$\therefore \quad I = \frac{\pi}{2} e^{-s}$$

Thus $F_c\left\{\frac{1}{1+x^2}\right\} = I = \frac{\pi}{2} e^{-s}$

$$\frac{dI}{ds} = -\frac{\pi}{2} e^{-s}$$

using this in (2), we have

$$-\frac{\pi}{2} e^{-s} = F_s\left(-\frac{x}{1+x^2}\right)$$

$$\Rightarrow F_s\left(\frac{x}{1+x^2}\right) = \frac{\pi}{2} e^{-s}$$

Now by Parseval's identity

$$\frac{2}{\pi}\int_0^\infty |F_s(s)|^2 ds = \int_0^\infty |f(x)|^2 dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \left(\frac{\pi}{2} e^{-s}\right)^2 ds = \int_0^\infty \frac{x^2}{(1+x^2)^2} dx$$

$$\Rightarrow \int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \cancel{\frac{2}{\pi}} \cdot \frac{\pi^2}{4}\left[\frac{e^{-2s}}{-2}\right]_0^\infty$$

$$= \frac{\pi}{2} \cdot \frac{1}{2}\left(-e^{-2s}\right)_0^\infty$$

$$= \frac{\pi}{4} \quad \text{Ans}$$

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Problem ① Use Parseval's identity, prove that

(i) $$\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

(ii) $$\int_0^\infty \frac{\sin(at)}{t(a^2+t^2)}\,dt = \frac{\pi}{2}\left(\frac{1-e^{-a^2}}{a^2}\right)$$

Ans (i) let $F(x) = e^{-ax}$, $G(x) = e^{-bx}$

$$f_c(p) = \int_0^\infty F(x)\cos(px)\,dx$$

$$= \int_0^\infty e^{-ax}\cos(px)\,dx = \frac{a}{a^2+p^2}$$

$$g_c(p) = \int_0^\infty G(x)\cos(px)\,dx$$

$$= \int_0^\infty e^{-bx}\cos(px)\,dx = \frac{b}{b^2+p^2}$$

By Parseval's identity for F.T., we have

$$\frac{2}{\pi}\int_0^\infty f_c(p)\,g_c(p)\,dp = \int_0^\infty F(x)\,G(x)\,dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+p^2}\cdot\frac{b}{b^2+p^2}\,dp = \int_0^\infty e^{-ax}\cdot e^{-bx}\,dx$$

$$\Rightarrow \int_0^\infty \frac{dp}{(a^2+p^2)(b^2+p^2)} = \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}\,dx$$

$$= \frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\Rightarrow \int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$ Proved

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ii) Let $F(x) = e^{-ax}$, then

$$f_c(p) = \frac{a}{a^2+p^2}$$, Prove as in (i)

Also let $G(x) = \begin{cases} 1, & 0<x<a \\ 0, & x>a \end{cases}$

Then $$g_c(p) = \int_0^\infty G(x)\cos(px)\,dx$$

$$= \int_0^a G(x)\cos(px)\,dx + \int_a^\infty G(x)\cos(px)\,dx$$

$$= \int_0^a 1\cdot\cos(px)\,dx + \int_a^\infty 0\cdot\cos(px)\,dx$$

$$= \left[\frac{\sin px}{p}\right]_0^a + 0 = \frac{\sin pa}{p}$$

Now $$\frac{2}{\pi}\int_0^\infty f_c(p)\,g_c(p)\,dp = \int_0^\infty F(x)\cdot G(x)\,dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+p^2}\cdot\frac{\sin(pa)}{p}\,dp = \int_0^\infty e^{-ax}\cdot G(x)\,dx$$

$$\Rightarrow \frac{2a}{\pi}\int_0^\infty \frac{\sin(pa)}{p(a^2+p^2)}\,dp = \int_0^a e^{-ax}G(x)\,dx + \int_a^\infty e^{-ax}G(x)\,dx$$

$$= \int_0^a e^{-ax}\cdot 1\,dx + \int_a^\infty e^{-ax}\cdot 0\,dx$$

$$= \left[\frac{e^{-ax}}{-a}\right]_0^a + 0 = \frac{1}{a}\left(1-e^{-a^2}\right)$$

$$\Rightarrow \int_0^\infty \frac{\sin(at)}{t(t^2+a^2)}\,dt = \frac{\pi}{2a^2}\left(1-e^{-a^2}\right)$$

Ans

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