Fourier Transform (8)

Problem (16) Find $f(x)$, if its fourier cosine transform is $\frac{1}{1+s^2}$, Or, If $\bar{f_c}(s) = F_c\{f(x)\}$ then find $f(x)$.

Answer

$\because f(x) = F_c^{-1}\{\bar{f_c}(s)\} = F_c^{-1}\{\frac{1}{1+s^2}\}$.

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+s^2} \cdot \cos sx \, ds \quad \text{---} (1)$$

$$\Rightarrow \frac{df(x)}{dx} = \frac{2}{\pi}\int_0^\infty \frac{-\sin sx \cdot s}{1+s^2} ds \quad \text{---} (2)$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{s^2 \sin sx}{s(1+s^2)} ds$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{(1+s^2-1)}{s(1+s^2)} \cdot \sin sx \, ds.$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s} ds + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)} ds$$

$$= -\frac{2}{\pi} \cdot \frac{\pi}{2} + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)} ds.$$

$$\Rightarrow \frac{df}{dx} = -1 + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)} ds \quad \text{---} (3)$$

$$\Rightarrow \frac{d^2f}{dx^2} = \frac{2}{\pi}\int_0^\infty \frac{s \cos sx}{s(1+s^2)} ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{1+s^2} \cdot \cos sx \, ds = f$$

$$\Rightarrow \frac{d^2f}{dx^2} - f = 0$$

Its complementry solution is

$$f = Ae^{-x} + Be^{x} \quad \text{——— (4)}$$

Putting $x=0$ in (1), we have

$$f(0) = \frac{2}{\pi}\int_0^\infty \frac{ds}{1+s^2} = \frac{2}{\pi}\left[\tan^{-1}s\right]_0^\infty = 1$$

Putting $x=0$ in (3), we have

$$\frac{df}{dx} = -1$$

$\therefore$ $f = 1$, when $x=0$ ——— (5)

$\frac{df}{dx} = -1$, when $x=0$ ——— (6)

Now by (4), we have

$$A + B = 1 \; [\text{by (5) \& (6)}] \quad \text{——— (7)}$$

Diffing (4) w.r.t. x

$$\frac{df}{dx} = -Ae^{-x} + Be^{x}$$

applying (5) & (6), we have

$$-A + B = -1 \quad \text{——— (8)}$$

Solving (7) & (8), we have, $B=0$ and $A=1$

Now by (4), solution is

$$f = e^{-x}$$ Ans

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Fourier transform (19)

Problem (17) Find the Fourier Sine and Cosine transform of the function x^{m-1}.

Answer: Let $f(x) = x^{m-1}$, $F_c\{f(x)\} = \bar{f}_c(s)$ and $F_s\{f(x)\} = \bar{f}_s(s)$.

Now, $$\bar{f}_c(s) = \int_0^\infty f(x)\cos sx\,dx = \int_0^\infty x^{m-1}\cos(sx)\,dx \quad (1)$$

and $$\bar{f}_s(s) = \int_0^\infty f(x)\sin sx\,dx = \int_0^\infty x^{m-1}\sin(sx)\,dx \quad (2)$$

By definition of Gamma function

$$\Gamma(m) = \int_0^\infty e^{-x} \cdot x^{m-1}\,dx$$

Putting $x = isy$ and noting $i = e^{i\pi/2}$

Now we get $$\Gamma(m) = \int_0^\infty e^{-isy}\,(isy)^{m-1}\,(is\,dy)$$

$$\Rightarrow \Gamma(m) = \int_0^\infty e^{-isy}\,(is)^m\,dy$$

$$\Rightarrow \frac{\Gamma(m)}{s^m} = \int_0^\infty e^{-isy} \cdot y^{m-1} \cdot i^m\,dy$$

$$= \left(e^{\frac{i\pi}{2}}\right)^m \cdot \int_0^\infty e^{-isy}\,y^{m-1}\,dy$$

$$= \left(e^{\frac{i\pi}{2}}\right)^m \int_0^\infty e^{-isx}\,x^{m-1}\,dx$$

$$\Rightarrow \int_0^\infty e^{-isx} \cdot x^{m-1}\,dx = \frac{\Gamma(m)}{s^m} \cdot e^{-\frac{im\pi}{2}}$$

$$\Rightarrow \int_0^\infty (\cos sx - i\sin sx)\,x^{m-1}\,dx = \frac{\Gamma(m)}{s^m}\left\{\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right\}$$

Equating real and imaginary parts we have

$$\int_0^\infty x^{m-1}\cos(sx)\,dx = \frac{\Gamma(m)}{s^m}\cos\left(\frac{m\pi}{2}\right)$$

and $\displaystyle\int_0^\infty x^{m-1}\sin(sx)\,dx = \frac{\Gamma(m)}{s^m}\sin\left(\frac{m\pi}{2}\right)$

$$\Rightarrow \bar{f}_s(s) = \frac{\Gamma(m)}{s^m}\sin\left(\frac{m\pi}{2}\right)$$

and $\displaystyle\bar{f}_c(s) = \frac{\Gamma(m)}{s^m}\cos\left(\frac{m\pi}{2}\right)$ Answer.

Problem 18 Find the sine and cosine transforms of $\dfrac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}$.

Answer Let $f(x) = \dfrac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}$

(i) We have to find $F_s\{f(x)\} = \bar{f}_s(s)$.

$$\bar{f}_s(s) = \int_0^\infty f(x)\sin sx\,dx$$

$$= \int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}\cdot\sin(sx)\,dx$$

$$= \int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}\cdot\frac{e^{isx}-e^{-isx}}{2i}\,dx$$

$$= \frac{1}{2i}\int_0^\infty \left[\frac{e^{(a+is)x}-e^{(a-is)x}+e^{-(a-is)x}-e^{-(a+is)x}}{e^{\pi x}-e^{-\pi x}}\right]dx$$

$$= \frac{1}{2i}\int_0^{\infty}\left[\frac{e^{(a+is)x} - e^{-(a+is)x}}{e^{\pi x} - e^{-\pi x}} - \frac{e^{(a-is)x} - e^{-(a-is)x}}{e^{\pi x} - e^{-\pi x}}\right]dx$$

$$= \frac{1}{2i}\left[\frac{1}{2}\tan\left(\frac{a+is}{2}\right) - \frac{1}{2}\tan\left(\frac{a-is}{2}\right)\right] \quad \{\text{from definite integral}\}$$

$$= \frac{1}{4i}\left[\frac{\sin\left(\frac{a+is}{2}\right)}{\cos\left(\frac{a+is}{2}\right)} - \frac{\sin\left(\frac{a-is}{2}\right)}{\cos\left(\frac{a-is}{2}\right)}\right]$$

$$= \frac{1}{4i}\left[\frac{\sin\left(\frac{a+is}{2}\right)\cos\left(\frac{a-is}{2}\right) - \cos\left(\frac{a+is}{2}\right)\sin\left(\frac{a-is}{2}\right)}{\cos\left(\frac{a+is}{2}\right)\cos\left(\frac{a-is}{2}\right)}\right]$$

$$= \frac{1}{2i}\left[\frac{\sin\left(\frac{a+is}{2} - \frac{a-is}{2}\right)}{2\cos\left(\frac{a+is}{2}\right)\cos\left(\frac{a-is}{2}\right)}\right]$$

$$= \frac{1}{2i}\left[\frac{\sin(is)}{\cos a + \cos(is)}\right] = \frac{\sinh(s)}{2\{\cos a + \cosh(s)\}}$$

$$= \frac{e^{s} - e^{-s}}{2\left(\cos a + \frac{e^{s} + e^{-s}}{2}\right)}$$

$$= \frac{e^{s} - e^{-s}}{2\left(2\cos a + e^{s} + e^{-s}\right)} \quad \underline{\text{Ans}}$$

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(ii) we have to find $F_c\{f(x)\} = \bar{f}_c(s)$.

Now $\bar{f}_c(s) = \int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cdot \cos sx \, dx$

$$= \int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cdot \frac{e^{isx} + e^{-isx}}{2} \, dx$$

$$= \frac{1}{2}\int_0^\infty \left[\frac{e^{(a+is)x} + e^{-(a+is)x}}{e^{\pi x} - e^{-\pi x}} + \frac{e^{(a-is)x} + e^{-(a-is)x}}{e^{\pi x} - e^{-\pi x}} \right.$$

$$= \frac{1}{2}\left[\frac{1}{2}\sec\left(\frac{a+is}{2}\right) + \frac{1}{2}\sec\left(\frac{a-is}{2}\right)\right]$$

$$= \frac{1}{4} \cdot \frac{\cos\left(\frac{a-is}{2}\right) + \cos\left(\frac{a+is}{2}\right)}{\cos\left(\frac{a+is}{2}\right)\cos\left(\frac{a-is}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{2\cos\left(\frac{a}{2}\right)\cos\left(\frac{is}{2}\right)}{\cos a + \cos(is)} = \frac{1}{2} \cdot \frac{\cos\left(\frac{a}{2}\right)\cosh\left(\frac{s}{2}\right)}{\cos a + \cosh(s)}$$

$$= \frac{\cos\left(\frac{a}{2}\right) \cdot e^{\frac{s}{2}} + e^{-\frac{s}{2}}}{2\cos a + (e^{s} + e^{-s})}$$

Ans.