FOURIER TRANSFORM (FINITE) (ii)

Definition ① The Finite Fourier Sine transform of $F(x)$:
The finite fourier sine transform of $F(x)$, where $0<x<l$, is defined by

$$F_s\{F(x)\} = f_s(s) = \int_0^l F(x)\sin\frac{s\pi x}{l}\,dx,$$

where s is a positive integer. The function $F(x)$ is then called the inverse finite Fourier Sine transform of $f_s(s)$ and is given by

$$F_s^{-1}\{f_s(s)\} = F(x) = \frac{2}{l}\sum_{s=1}^{\infty} f_s(s)\sin\frac{s\pi x}{l}.$$

Definition ② The Finite Fourier Cosine transform of $F(x)$:
The finite fourier Cosine transform of $F(x)$, where $0<x<l$, is defined by

$$F_c\{F(x)\} = f_c(s) = \int_0^l F(x)\cos\frac{s\pi x}{l}\cdot dx,$$

where s is a positive or zero integer. The function $F(x)$ is then called the inverse finite cosine transform of $f_c(s)$ and is given by

$$F_c^{-1}\{f_c(s)\} = F(x) = \frac{1}{l}f_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} f_c(s)\cos\frac{s\pi x}{l}.$$

Theorem ① Fourier integral Theorem

Statement: If $f(x)$ satisfies the following conditions (i) $f(x)$ satisfies the Dirichlet Conditions in every interval $-l\le x\le l$. (ii) $\int_{-\infty}^{\infty}|f(x)|\,dx$ converges ie $f(x)$ is absolutely integrable in the interval $-\infty<x<\infty$, then

$$f(x) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\int_{t=-\infty}^{\infty} f(t)\cos s(x-t)\,ds\,dt$$

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The integral on R.H.S. is called Fourier integral or Fourier integral expansion of $f(x)$.

Proof: let us consider a function $f(x)$ satisfying Dirichlet's condition in every interval $(-c, c)$, however large. Also let

$$\int_{-\infty}^{\infty} |f(x)|\,dx \text{ is convergent. } \quad (1)$$

Then in the interval $(-c, c)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c} \right) \quad (2)$$

where $a_n = \frac{1}{c}\int_{-c}^{c} f(t) \cos\frac{n\pi t}{c}\,dt$, $n = 0, 1, 2, \ldots$

and $b_n = \frac{1}{c}\int_{-c}^{c} f(t) \sin\frac{n\pi t}{c}\,dt$, $n = 1, 2, 3, \ldots$

Putting the values of a_n and b_n in (2),

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt +$$

$$\frac{1}{c}\sum_{n=1}^{\infty} \int_{-c}^{c} f(t) \left\{ \cos\frac{n\pi t}{c} \cos\frac{n\pi x}{c} + \sin\frac{n\pi t}{c} \sin\frac{n\pi x}{c} \right\} dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c} f(t) \cos\frac{n\pi}{c}(t-x)\,dt.$$

Making use of the fact that $f(x)$ is uniformly convergent in the closed interval $-c \leq x \leq c$, then we set

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\int_{-c}^{c} f(t) \left[\sum_{n=1}^{\infty} \cos\frac{n\pi(t-x)}{c}\,dt \right]$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t) \left[1 + \lim_{n\to\infty} \sum_{r=1}^{n} 2\cos\frac{r\pi(t-x)}{c} \right] dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c}\left[1 + \lim_{n\to\infty}\sum_{r=1}^{n}\left\{\cos\frac{r\pi(t-x)}{c} + \cos\frac{-r\pi(t-x)}{c}\right\}\right]f(t)\,dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\left[1 + \lim_{n\to\infty}\sum_{r=-n}^{n}\cos\frac{r\pi(t-x)}{c}\right]dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\lim_{n\to\infty}\sum_{r=-n}^{n}\frac{1}{c/\pi}\cos\frac{r}{c/\pi}(t-x)\right]dt$$

Making use of definition of integral as a limit of sum, we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{2\pi}\int_{-c}^{c}\left[\int_{-\infty}^{\infty}\cos u(t-x)\,du\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{2\pi}\int_{-c}^{c} f(t)\,dt\int_{-\infty}^{\infty}\cos u(t-x)\,du$$

Making $c \to \infty$ and using (1)

$$f(x) = 0 + \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty}\cos u(t-x)\,du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\,dt\,du$$

Finally if $-\infty < x < \infty$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(t)\cos s(t-x)\,ds\,dt$$

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