Thm: A polynomial of degree n over a field can have at most n roots in any extension field.

Proof We shall prove this by induction on n i.e, degree of polynomial $p(x)$.

Let $p(x) = a_0x + a_1$ i.e. $p(x)$ is polynomial of degree 1.

Obviously $a_0 \neq 0$.

Let a be a root of $p(x)$ in some extension field

$\Rightarrow p(a) = 0$

$\Rightarrow a_0a + a_1 = 0$

$\Rightarrow a = -\frac{a_1}{a_0}$ which is a unique element of F.

Let us assume that the assumption is true for all polynomials of degree less than n. Let $p(x)$ be a polynomial of degree n.

Let K be any extension of F. If $p(x)$ has no root in K, then the theorem is true because the number of roots of $p(x)$ in K is zero which is definitely at most n. So let $p(x)$ has atleast one root, say $a \in K$. Let the multiplicity of this root be m.

$\Rightarrow (x-a)^m$ is divisor of $p(x)$

$\Rightarrow$ degree of $(x-a)^m \leq$ degree of $p(x)$

$\Rightarrow m \leq n$

Since in $K[x]$ $(x-a)^m$ is divisor of $p(x)$

So let $p(x) = (x-a)^m \cdot q(x)$ where $q(x) \in K[x]$

So degree of $q(x) = \deg p(x) - \deg (x-a)^m$

$= n - m < n \quad (\because 1 \le m \le n$

Now a is a root of $p(x)$ of multiplicity m so $(x-a)^{m+1}$ is not a divisor of $p(x) = (x-a)^m q(x)$

$\Rightarrow$ $(x-a)$ is not a divisor of $q(x)$

$\Rightarrow$ a is not a root of $q(x)$

Let $b \neq a$ be a root of $p(x)$ in K, then on putting $x = b$ in $p(x)$ we get

$$p(b) = 0$$

$$\Rightarrow (b-a)^m \cdot q(b) = 0$$

Since K is a field and $b - a \neq 0$ ie $(b-a)^m \neq 0$

$$\Rightarrow q(b) = 0$$

$\therefore$ b is root of $q(x)$ in K

Now any root of $p(x)$ other than a is a root of $q(x)$ in K whose degree is $n-m$ which is less than m. By induction hypothesis $q(x)$ has at most $n-m$ roots in K. and no of roots is equal to a

$\Rightarrow$ $q(x)$ has at most $n-m$ roots other than a

$\Rightarrow$ $p(x)$ has at most $n-m$ roots other than a

$\Rightarrow$ $p(x)$ has at most $(n-m)+m$ roots in K.

(Adding multiplicity of root a)

ie $p(x)$ has at most n roots in K

Hence by induction, theorem is proved