FOURIER TRANSFORM (12)

Theorem 2 Different forms of Fourier Integral Formulae:

(i) $$f(x) = \frac{1}{\pi}\int_{s=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)\cos s(t-x)\,dt.$$

Proof: By Fourier's integral formula

$$f(x) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\cos s(t-x)\,ds\int_{t=-\infty}^{\infty} f(t)\,dt$$

$$= \frac{2}{2\pi}\int_{0}^{\infty}\cos s(t-x)\,ds\int_{t=-\infty}^{\infty} f(t)\,dt \quad \left(\text{By Property of D.I}\right)$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos s(t-x)\,ds\,dt.$$

(ii) Cosine form:

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(t)\cos st\cos sx\,ds\,dt.$$

Proof: By Fourier's integral formula

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos s(t-x)\,ds\,dt.$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos s(t-x)\,ds\,dt \quad [\text{by (i)}]$$

Taking $A(s) = \int_{-\infty}^{\infty} f(t)\cos st\,dt$

and $B(s) = \int_{-\infty}^{\infty} f(t)\sin st\,dt$.

Then we have

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\left(\cos st\cos sx + \sin st\sin sx\right)ds\,dt$$

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(s).\cos sx + B(s).\sin sx]\,ds \quad \text{--- (i)}$$

Let $f(t)$ be an even function of t, so that $f(-t) = f(t)$. Then $f(t)\cos st$ is even and $f(t)\sin st$ is odd function of t.

Then $A(s) = 2\int_0^\infty f(t)\cos st\,dt$, $B(s) = 0$

Putting these values in (1), we have

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(s).\cos sx + 0.\sin sx]\,ds$$

$$= \frac{1}{\pi}\int_0^\infty A(s)\cos sx\,ds$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos st\cos sx\,ds.dt$$

$$= \frac{1}{\pi}\int_0^\infty 2\int_0^\infty f(t)\cos st\cos sx\,ds.dt$$

$$= \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\cos st\cos sx\,ds.dt$$

(iii) Sine form:

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\sin st\sin sx\,ds.dt$$

Proof: Prove as in case (ii) upto (1), we have

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(s).\cos sx + B(s).\sin sx]\,ds \quad \text{--- (1)}$$

Let $f(t)$ be an odd function, then $f(-t) = -f(t)$. Then $f(t)\cos st$ and $f(t)\sin st$ are odd and even functions respectively

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Then $A(S) = 0$ and $B(S) = 2\int_0^\infty f(t) \sin st \, dt$.

Putting these values in ①, we have

$$f(s) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t) \sin st \cdot \sin sx \, ds \cdot dt$$

(iv) Exponential form: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{-ist} e^{isx} ds \cdot dt$.

Proof: By integral formula, we have

$$f(x) = \frac{1}{\pi}\int_{s=0}^{\infty} \int_{t=-\infty}^{\infty} f(t) \cos s(t-x) \, ds \cdot dt \quad [\text{by (i)}]$$

$$= \frac{1}{\pi}\int_{s=0}^{\infty} \int_{t=-\infty}^{\infty} f(t) \left[\frac{e^{-is(t-x)} + e^{is(t-x)}}{2}\right] ds \cdot dt$$

$$= \frac{1}{2\pi}\int_0^{\infty} \int_{-\infty}^{\infty} f(t) \left[e^{-ist} \cdot e^{isx} + e^{ist} \cdot e^{-isx}\right] ds \cdot dt$$

$$\frac{1}{2\pi}\int_0^{\infty} \int_{-\infty}^{\infty} f(t) e^{-ist} e^{isx} ds\,dt + \frac{1}{2\pi}\int_0^{\infty} \int_{-\infty}^{\infty} f(t) e^{ist} \cdot e^{-isx} ds\,dt$$

Putting $s = -s'$ in second integral

$$f(x) = \frac{1}{2\pi}\int_0^{\infty} \int_{-\infty}^{\infty} f(t) e^{isx} \cdot e^{-ist} ds \cdot dt +$$

$$\frac{1}{2\pi}\int_0^{-\infty} \int_{-\infty}^{\infty} f(t) e^{-is't} \cdot e^{is'x} (-ds') \cdot dt$$

$$f(x) = \frac{1}{2\pi}\int_0^{\infty} \int_{-\infty}^{\infty} f(t) e^{-ist} e^{isx} ds\,dt + \frac{1}{2\pi}\int_{-\infty}^{0} \int_{-\infty}^{\infty} f(t) \cdot e^{-ist} e^{isx} ds\,dt$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot e^{-ist} \cdot e^{isx} ds \cdot dt$$

(on dropping primes)

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