[illegible]

$F(x)$ is of b.v. in $[a,b]$ iff $V_F(b)$ is bounded in $[a,b]$

Now. $V_F(b) = \sup \sum_{k=0}^{n-1} |F(x_{k+1}) - F(x_k)|$, supremum being taken over all partition of $[a,b]$.

$$= \sup \sum_{k=0}^{n-1} \left| \int_a^{x_{k+1}} f(t)\,d\alpha(t) - \int_a^{x_k} f(t)\,d\alpha(t) \right|$$

$$= \sup \sum_{k=0}^{n-1} \left| \int_a^{x_{k+1}} f(t)\,d\alpha(t) + \int_{x_k}^{a} f(t)\,d\alpha(t) \right|$$

$$= \sup \sum_{k=0}^{n-1} \left| \int_{x_k}^{x_{k+1}} f(t)\,d\alpha(t) \right|$$

$$\leq \sup \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |f(t)|\,|d\alpha(t)|$$

$$\leq \sup M \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |d\alpha(t)|$$

{∵ $f(x)$ is continuous in $[a,b] \Rightarrow f(x)$ is bounded in $[a,b]$

$\Rightarrow f(x) \leq M$ (a finite quantity)}

$$= \sup M \int_a^b |d\alpha(t)|$$

$$\leq \sup M \int_a^b dV_\alpha(t)$$

$$= \sup M \left[V_\alpha(t) \right]_a^b$$

$$= \sup M \left[V_\alpha(b) - V_\alpha(a) \right]$$

$$= M\, V_\alpha(b) \qquad \{∵ V_\alpha(a) = 0\}$$

$V_F(b)$ = finite ; since $\alpha(x)$ is of b.v. in $[a,b] \Rightarrow V_\alpha(b)$ is bounded

$\Rightarrow V_F(b) \leq$ a finite quantity

$\Rightarrow V_F(b)$ is bounded

Hence, $F(x)$ is of bounded variation in $[a,b]$

Ex) 3

To prove $F(x) = \dfrac{F(x+) + F(x-)}{2}$ for $a < x < b$

Now $F(x+) = \lim\limits_{h\to 0} F(x+h) = \lim\limits_{h\to 0} \int_a^{x+h} f(t)\,d\alpha(t)$

[Note: To prove $F(x) = \dfrac{F(x+) + F(x-)}{2}$
given $F(x+) + F(x-) = 2F(x)$ if $F(x+) - F(x) = F(x) - F(x-)$]

$$\therefore F(x+) - F(x) = \lim_{h\to 0} \left[\int_a^{x+h} f(t)\,d\alpha(t) - \int_a^{x} f(t)\,d\alpha(t)\right]$$

$$= \lim_{h\to 0} \left\{\int_a^{x+h} f(t)\,d\alpha(t) - \int_a^{x} f(t)\,d\alpha(t)\right\}$$

$$= \lim_{h\to 0} \left\{\int_a^{x+h} f(t)\,d\alpha(t) + \int_x^{a} f(t)\,d\alpha(t)\right\}$$

$$= \lim_{h\to 0} \int_x^{x+h} f(t)\,d\alpha(t)$$

Applying Mean value theorem, we have

$$= \lim_{h\to 0} f(x+\theta h)\left[\alpha(x+h) - \alpha(x)\right], \text{ where } 0 < \theta < 1$$

$$F(x+) - F(x) = f(x)\left[\alpha(x+) - \alpha(x)\right] \quad \text{--- (1)}$$

[$\because f(x)$ is continuous in $[a,b]$, $\therefore \lim\limits_{h\to 0} f(x+\theta h) = f(x)$]

Again

$$F(x) - F(x-) = \int_a^{x} f(t)\,d\alpha(t) - \lim_{h\to 0} \int_a^{x-h} f(t)\,d\alpha(t)$$

$$= \lim_{h\to 0} \left\{\int_a^{x} f(t)\,d\alpha(t) - \int_a^{x-h} f(t)\,d\alpha(t)\right\}$$

$$\Rightarrow F(x) - F(x-) = \lim_{h\to 0} \int_{x-h}^{x} f(x)\,d\alpha(t)$$

applying Mean value theorem, we have

$$F(x) - F(x-) = \lim_{h\to 0} f(x-h+\theta h)\left[\alpha(x) - \alpha(x-h)\right];\ 0$$

$$\Rightarrow F(x) - F(x-) = f(x)\left[\alpha(x) - \alpha(x-)\right] \quad \text{--- (2)}$$

Subtracting 2 [illegible] from [illegible] right

$$F(x+) + F(x-) - 2F(x) = f(x)[\alpha(x+) - \alpha(x) - \alpha(x) + \alpha(x-)]$$

$$= f(x)[\alpha(x+) + \alpha(x-) - 2\alpha(x)]$$

$$\Rightarrow \frac{F(x+) + F(x-)}{2} - F(x) = f(x)\left[\frac{\alpha(x+) + \alpha(x-)}{2} - \alpha(x)\right]$$

$$= f(x)[\alpha(x) - \alpha(x)] \quad \left\{ \because \alpha(x) \text{ is normalised } \alpha(x) = \frac{\alpha(x+) + \alpha(x-)}{2} \right.$$

$$\Rightarrow \frac{F(x+) + F(x-)}{2} - F(x) = 0$$

$$\Rightarrow F(x) = \frac{F(x+) + F(x-)}{2}$$

Thus all these properties on $F(x)$ proves that $F(x)$ is a normalised function

---

Theorem 8(c) :- If $\alpha(x)$ is a normalised function in $a \leq x \leq R$ $\forall R > a$ and if $\lim_{x\to\infty} \alpha(x) = A$ $(x \in E)$, x becoming infinite through the set E, of points of continuity of $\alpha(x)$, then $\lim_{x\to\infty} \alpha(x) = A$ $(a < x < R$

Proof :- Since $\alpha(x)$ is a normalised function in $[a, R]$

$\therefore$ for $x \in]a, R[$

$$\alpha(x) = \frac{\alpha(x+) + \alpha(x-)}{2}$$

Given that $\lim_{x\to\infty} \alpha(x) - A$ $(x \in E)$

$$\Rightarrow |\alpha(x) - A| < \epsilon \quad \text{for } x > x_0(\epsilon)$$

$$\Rightarrow |\alpha(x+) - A| < \epsilon \quad \text{——} \quad (1)$$

and $$|\alpha(x-) - A| < \epsilon \quad \text{——} \quad (2)$$

Since $x \in E \Rightarrow x$ is a point of continuity $\alpha(x)$ (given)

$$\Rightarrow \alpha(x+) = \alpha(x-) = \alpha(x)$$

Now, for any x satisfying $a < x < R$;

$$\therefore |\alpha(x) - A| = \left|\frac{\alpha(x+) + \alpha(x-)}{2} - A\right|$$

$$= \frac{1}{2}|\alpha(x+) + \alpha(x-) - 2A|$$

$$= \frac{1}{2}\left| \alpha(x+) - A + \alpha(x-) - A \right|$$

$$\leq \frac{1}{2}\left\{ |\alpha(x+) - A| + |\alpha(x-) - A| \right\}$$

$$< \frac{1}{2}(\epsilon + \epsilon), \; x > x_0(\epsilon) \text{ [by eq (1) and (2)]}$$

$$= \frac{1}{2} \cdot 2\epsilon = \epsilon$$

$\therefore |\alpha(x) - A| < \epsilon$, for $x > x_0(\epsilon)$ and $a < x < R$

$\Rightarrow \lim_{x \to \infty} \alpha(x) = A$, for any x satisfying $a < x < R$

Theorem 9: If $f(x)$ is continuous in $a \leq x \leq \infty$, if $\alpha(x)$ is of b.v. in $a \leq x \leq R$, $\forall$ $R > a$ and if $\alpha^*(x)$ is the normalised functions of $\alpha(x)$, then $\int_a^\infty f(x)\,d\alpha(x) = \int_a^\infty f(x)\,d\alpha^*(x)$ provided the 1st integral converges.

[R.U. 73, 82]

Proof Since $\alpha(x)$ is of b.v. in $a \leq x \leq R$
and $\alpha^*(x)$ is a normalised function of $\alpha(x)$.
$\therefore$ $\alpha^*(x)$ is of b.v. in $a \leq x \leq R$.

$$\therefore \int_a^R f(x)\,d\alpha(x) = \int_a^R f(x)\,d\alpha^*(x), \; \forall R > a$$

$$\Rightarrow \lim_{R \to \infty} \int_a^R f(x)\,d\alpha(x) = \lim_{R \to \infty} \int_a^R f(x)\,d\alpha^*(x),$$

provided both the limits exist.

If $\int_0^\infty f(x)\,d\alpha(x)$ is convergent

$\Rightarrow \lim_{R \to \infty} \int_a^R f(x)\,d\alpha(x)$ exists

$\Rightarrow \int_a^\infty f(x)\,d\alpha(x)$ exists — (1)

Now, we put $F(R) = \int_a^R f(x)\, d\alpha^*(x)$

Since $f(x)$ is continuous and $\alpha^*(x)$ is of b.v. in $[a, R]$

$\Rightarrow$ $F(R)$ is of b.v. in $[a, R]$

$\Rightarrow$ $F(R)$ is bounded in $[a, R]$

$\Rightarrow$ $|F(R)| < M$, $\forall R$, where M is finite

$\because$ $F(R) \leq |F(R)| < M$

$\Rightarrow$ $F(R) < M$ $\forall R$.

$\Rightarrow$ $\lim_{R \to \infty} F(R) < M$

$\Rightarrow$ $\lim_{R \to \infty} F(R)$ exists $= A$ (say)

$\therefore$ $\lim_{R \to \infty} \int_a^R f(x)\, d\alpha^*(x)$ exists and $= A$

$\Rightarrow$ $\int_a^\infty f(x)\, d\alpha^*(x)$ exists — ⟨II⟩

Hence from ⟨I⟩ and ⟨II⟩, we can write

$$\lim_{R \to \infty} \int_a^R f(x)\, d\alpha(x) = \lim_{R \to \infty} \int_a^R f(x)\, d\alpha^*(x)$$

$$\Rightarrow \int_a^\infty f(x)\, d\alpha(x) = \int_a^\infty f(x)\, d\alpha^*(x)$$

Laws of Mean

1st Mean value theorem :- [R.U.-73]

⟨a⟩ :- Statement :- If $f(x)$ is real contin