CHAPTER-3

Laplace Transform → (1) Laplace Stieltjes Integral

Laplace Transform → (2) Laplace Integral

(1) Define Laplace-Stieltjes integral.

If a function $\alpha(t)$ of b.v. be a complex function i.e. if we can write $\alpha(t)$ as

$$\alpha(t) = \alpha'(t) + i\alpha''(t),$$ where $\alpha'(t)$ and $\alpha''(t)$ are real functions

then the improper integral $\int^{\infty} e^{-st}\, d\alpha(t)$ is a Laplace-Stieltjes integral or L-S Transform where s is a complex variable defined by

$$s = \sigma + i\tau$$

Now, $\alpha(t)$ is of b.v. in $0 \le t \le R$, $\forall$ +ve R, then the L-S integral defined by

$$\int_0^{\infty} e^{-st}\, d\alpha(t) = \lim_{R \to \infty} \int_0^R e^{-st}\, d\alpha(t) \quad \text{——} \ (1)$$

The R.H.S of (1) is called the cauchy value at its upper limit.

If $\alpha(t)$ is of b.v. in $\epsilon < t < R$, $\forall$ +ve ϵ and R then L-S integral defined by

$$\int_0^{\infty} e^{-st}\, d\alpha(t) = \lim_{\substack{\epsilon \to 0 \\ R \to \infty}} \int_{0+\epsilon=\epsilon}^{R} e^{-st}\, d\alpha(t) = \lim_{R \to \infty} \int_{0+}^{R} e^{-st}\, d\alpha(t)$$

If the limit in (1) exists, we say L-S integral is cgt.

Also, if the limit in (2) exists, we say L-S integral is cgt.

$\Rightarrow \int_{0+}^{\infty} e^{-st}\, t \sin\frac{1}{t}\, dt$ = a finite quantity

$\therefore \int_{0+}^{\infty} e^{-st}\, d\alpha(t)$ exists when $\alpha(t)$ is not of b.v.
(e, R

<u>Note</u>:- [Take the example of theorem (A) previous]

Q Define Laplace integral

<u>Ans</u> If the complex function $\alpha(t)$ be the integral of
a function $\varphi(t)$ in $[0, t]$
i.e. if $\alpha(t) = \int_0^t \varphi(t)\, dt$

or, if $\alpha(t)$ be absolutely continuous complex f
then
$$\int_0^{\infty} e^{-st}\, d\alpha(t) = \int_0^{\infty} e^{-st}\, \varphi(t)\, dt$$

And the integral $\int_0^{\infty} e^{-st}\, \varphi(t)\, dt$ is called the
integral

Thus, the Laplace-integral is a p
type of L-S integral
Hence the improper integral $\int_0^{\infty} e^{-st}\, \varphi(t)\, dt$,
$\int_0^t \varphi(t)\, dt$ is a absolutely continuous
function, is called the Laplace-integral.

Theorem (1) If $\underset{0\le u<\infty}{\text{lub}} \left|\int_0^u e^{-s_0 t}\, d\alpha(t)\right| = M < \infty$, then $\int_0^\infty e^{-st}\, d\alpha(t)$ converges, $\forall s$ for which $\sigma > \sigma_0$ and

$$\int_0^\infty e^{-st}\, d\alpha(t) = (s-s_0)\int_0^\infty e^{-(s-s_0)t}\, \beta(t)\, dt \quad \text{— (1)}, \text{ where}$$

$$\beta(u) = \int_0^u e^{-s_0 t}\, d\alpha(t), \quad u \ge 0$$

The integral on R.H.S of (1) converging absolutely

Proof :- We have

$$\beta(u) = \int_0^u e^{-s_0 t}\, d\alpha(t) \;:\; u \ge 0$$

So that $\underset{0\le u<\infty}{\text{lub}} |\beta(u)| = M$

$\Rightarrow |\beta(u)| \le M$

$\forall u$ in $0 \le u < \infty$

$$\Rightarrow \beta(t) = \int_0^t e^{-s_0 t}\, d\alpha(t)$$

$$\Rightarrow d\{\beta(t)\} = e^{-s_0 t}\, d\alpha(t)$$

$$\Rightarrow d\alpha(t) = e^{s_0 t}\, d\beta(t)$$

$$\Rightarrow \int_0^\infty e^{-st}\, d\alpha(t) = \int_0^\infty e^{-(s-s_0)t}\, d\beta(t)$$

$$= \lim_{t\to\infty} \left[e^{-(s-s_0)t}\beta(t)\right]_0^t + (s-s_0)\int_0^\infty e^{-(s-s_0)t}\, \beta(t\ldots$$

(Int. by parts)

$$= \lim_{t\to\infty} \left[e^{-(s-s_0)t}\beta(t)\right] - \beta(0) + (s-s_0)\int_0^\infty e^{-(s-s_0)t}\, \beta(t\ldots$$

Now, $\beta(0) = \int_0^0 e^{-s_0 t}\, d\alpha(t) = 0$ — (2)

and

$$\left|e^{-(s-s_0)t}\beta(t)\right| = \left|e^{-(s-s_0)t}\right| \cdot |\beta(t)| = e^{-(\sigma-\sigma_0)t}\, |\beta(t\ldots$$

(where $s = \sigma + i\tau$ and $s_0 = \sigma_0 + i\tau_0$)

$$\le M e^{-(\sigma-\sigma_0)t}$$

, since by ~~hyp~~ hypothesis $\underset{0\le t<\infty}{\text{lub}} |\beta(t)| = M$

$$\Rightarrow |\beta(t)| \le M \quad \forall t$$

~~Also~~ And also we have seen that R.H.S integral of eqn.(1) i.e. $\int_0^\infty e^{-(s-s_0)t} \beta(t)\,dt$ converges absolutely

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[Coro]llary: If the integral $\int_0^\infty e^{-st}\,d\alpha(t)$ is convergent for $s = s_0 = \sigma_0 + i\tau_0$, it converges for all $s = \sigma + i\tau$ for which $\sigma > \sigma_0$ (or it converges uniformly for which $\sigma > \sigma_0$)

[Proo]f: Given $\int_0^\infty e^{-st}\,d\alpha(t)$ is convergent for $s = \sigma_0 + i\tau_0 = s_0$ (say)

$\Rightarrow \int_0^\infty e^{-s_0 t}\,d\alpha(t)$ is convergent

$\Rightarrow$ each of $\int_0^u e^{-s_0 t}\,d\alpha(t)$ exist where $0 \leq u \leq \infty$

$\Rightarrow$ the set $\{\int_0^u e^{-s_0 t}\,d\alpha(t);\ 0 \leq u \leq \infty\}$ is bounded

$\Rightarrow |\int_0^u e^{-s_0 t}\,d\alpha(t)|$ is a finite number, $\forall$ u satisf $0 < u < \infty$.

$\underset{0 \leq u < \infty}{u.b.} |\int_0^u e^{-s_0 t}\,d\alpha(t)| = M$, a finite number

We put $\beta(u) = \int_0^u e^{-s_0 t}\,d\alpha(t)$

So that $\underset{0 \leq u < \infty}{u.b.} |\beta(u)| = M$

$\Rightarrow |\beta(u)| \leq M$, $\forall$ u in $0 \leq u < \infty$

$\therefore \beta(t) = \int_0^t e^{-s_0 t}\,d\alpha(t)$

$\Rightarrow d\beta(t) = e^{-s_0 t}\,d\alpha(t)$

$\Rightarrow d\alpha(t) = e^{s_0 t}\,d\beta(t)$

(30)

$$\Rightarrow \int_0^\infty e^{-st}\, d\alpha(t) = \int_0^\infty e^{-(s-s_0)t}\, d\beta(t)$$

$$= \lim_{t\to\infty}\left[e^{-(s-s_0)t}\beta(t)\right]_0^t + (s-s_0)\int_0^\infty e^{-(s-s_0)t}\beta(t)\,dt$$

(Int by parts)

$$= \lim_{t\to\infty} e^{-(s-s_0)t}\beta(t) - \beta(0) + (s-s_0)\int_0^\infty e^{-(s-s_0)t}\beta(t)\,dt \quad \text{--- (1)}$$

Now, $\beta(0) = \int_0^0 e^{-s_0 t}\, d\alpha(t) = 0$ ——— (2)

$$\therefore \left| e^{-(s-s_0)t}\beta(t)\right| = \left| e^{-(s-s_0)t}\right| \cdot |\beta(t)|$$

$$= e^{-(\sigma-\sigma_0)t} \cdot |\beta(t)| \qquad \because s = \sigma + i\tau,\ s_0 = \sigma_0 + i\tau_0$$

$$\le e^{-(\sigma-\sigma_0)t} \cdot M \qquad \{\because |\beta(t)| \le M\}$$

$$\Rightarrow \therefore \left|\lim_{t\to\infty} e^{-(s-s_0)t}\beta(t)\right| \le M \lim_{t\to\infty} e^{-(\sigma-\sigma_0)t}$$

$$\le 0 \quad \text{if } \sigma > \sigma_0$$

$$\Rightarrow \left|\lim_{t\to\infty} e^{-(s-s_0)t}\beta(t)\right| = 0 \quad \text{if } \sigma > \sigma_0 \quad \{\because |\,| < 0 \text{ is impo}$$

$$\Rightarrow \lim_{t\to\infty} e^{-(s-s_0)t}\beta(t) = 0 \quad \text{if } \sigma > \sigma_0 \quad \text{(3)}$$

Hence (1) becomes by using (2) and (3)

$$\int_0^\infty e^{-st}\, d\alpha(t) = (s-s_0)\int_0^\infty e^{-(s-s_0)t}\beta(t)\,dt \quad \text{--- (4) if}$$

$$\text{Now, } \int_0^\infty \left| e^{-(s-s_0)t}\beta(t)\,dt\right| = \int_0^\infty e^{-(\sigma-\sigma_0)t}\,|\beta(t)|\,dt$$

$$\le M\int_0^\infty e^{-(\sigma-\sigma_0)t}\,dt$$

$$= M\left[\frac{e^{-(\sigma-\sigma_0)t}}{-(\sigma-\sigma_0)}\right]_0^\infty$$