FOURIER TRANSFORM (15)

Problem (6) Express the function $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ as a fourier integral. Hence evaluate

$$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda}\, d\lambda.$$

Answer: By fourier integral formula

$$f(x) = \frac{1}{\pi} \int_{s=0}^{\infty} \int_{t=-\infty}^{\infty} f(t) \cos s(t-x)\, ds.\, dt.$$

$$= \frac{1}{\pi} \int_{s=0}^{\infty} \int_{t=-1}^{1} 1 . \cos s(t-x)\, ds . dt$$

$$= \frac{1}{\pi} \int_{s=0}^{\infty} \left[\frac{\sin s(t-x)}{s} \right]_{t=-1}^{1} ds$$

$$= \frac{1}{\pi} \int_{s=0}^{\infty} \left[\frac{\sin s(1-x) - \sin s(-1-x)}{s} \right] ds.$$

$$= \frac{1}{\pi} \int_{s=0}^{\infty} \frac{\sin s(1-x) + \sin s(1+x)}{s}\, ds.$$

$$= \frac{1}{\pi} \int_{s=0}^{\infty} \frac{2 \sin s \cos sx}{s}\, ds$$

$$= \frac{2}{\pi} \int_{s=0}^{\infty} \frac{\sin s \cos sx}{s}\, ds = \frac{2}{\pi} \int_0 \frac{\sin\lambda \cos\lambda x}{\lambda}\, d$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda}\, d\lambda \quad \text{--- (1)}$$

This is the representation of $f(x)$ as fourier integral.

Subhash
K.C.C

Second Part : By ①, we have

$$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

$$\Rightarrow \int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda = \begin{cases} \frac{\pi}{2}, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases}$$

Ans.

Problem ⑦ Find fourier transform of $f(x)$, defined by $f(x) = \begin{cases} 1, & |x| \leqslant a \\ 0, & |x| > a \end{cases}$

and hence prove that $\int_0^\infty \frac{\sin^2 ax}{x^2} dx = \frac{\pi a}{2}$

Answer First Part to find $F\{f\}$.

$$F\{f\} = \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx$$

$$= \int_{-\infty}^{-a} e^{-ipx} f(x)\, dx + \int_{-a}^{a} e^{-ipx} f(x)\, dx + \int_{a}^{\infty} e^{-ipx} f(x)$$

Put $x = -y$

$$= \int_{\infty}^{a} e^{ipy} f(-y)\, d(-y) + \int_{-a}^{a} e^{-ipx} \cdot 1\, dx + \int_{a}^{\infty} e^{-ipx} \cdot 0\, d$$

$$= \int_a^\infty e^{ipy} \cdot 0\, dy + \left[\frac{e^{-ipx}}{-ip}\right]_{-a}^{a} + 0$$

$$= \frac{e^{ipa} - e^{-ipa}}{ip} = \frac{2}{p} \cdot \frac{e^{ipa} - e^{-ipa}}{2i}$$

$$= \frac{2}{p} \sin pa = \bar{f}(p)$$

Ans

Sattar
K.C.C.

Second Part using Parseval's identity for fourier integral, we get

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{F}(p)|^2 dp$$

$$\int_{-a}^{a} 1^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{2}{p}\sin pa\right)^2 dp$$

$$\Rightarrow [x]_{-a}^{a} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{p^2}\sin^2 pa \, dp$$

$$\Rightarrow [a+a] = \frac{2}{2\pi}\int_{0}^{\infty} \frac{4}{p^2}\sin^2 pa \, dp$$

$$\Rightarrow 2a = \frac{4}{\pi}\int_{0}^{\infty} \frac{\sin^2 pa}{p^2} dp$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin^2 pa}{p^2} dp = \frac{\pi a}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin^2 (ax)}{x^2} dx = \frac{\pi a}{2}$$ Ans

Problem (8) using fourier integral, show that $e^{-ax} = \frac{2a}{\pi}\int_{0}^{\infty} \frac{\cos \lambda x}{\lambda^2 + a^2} d\lambda, a>0, x\geq 0$

Answer By fourier Cosine integral formula

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(t)\cos st \cos sx \, ds \, dt.$$

taking $f(t) = e^{-at}$ we have

$$e^{-ax} = \frac{2}{\pi}\int_{0}^{\infty} \cos sx \, ds . \int_{0}^{\infty} e^{-at}\cos st \, dt$$

Sattesh K.C.C.

$$\Rightarrow e^{-ax} = \frac{2}{\pi}\int_0^\infty \cos sx \, ds \cdot \frac{a}{s^2+a^2}$$

$$= \frac{2a}{\pi}\int_0^\infty \frac{\cos sx}{s^2+a^2} ds = \frac{2a}{\pi}\int_0^\infty \frac{\cos \lambda x}{\lambda^2+a^2} d\lambda.$$

Ans

Problem (9) using fourier integral formula

show that $\frac{\pi}{2} e^{-x} = \int_0^\infty \frac{\cos \lambda x}{\lambda^2+1} d\lambda.$

Answer solve problem (8) completely and put $a=1$.

Problem (10) If $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

and $\bar{f}(s) = \frac{2\sin sa}{s}$, $s \neq 0$, then prove that

$$\int_0^\infty \frac{\sin^2 ax}{x^2} dx = \frac{\pi a}{2}.$$

Answer using Parseval's identity for F.T.

$$\int_{-\infty}^\infty |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^\infty |\bar{f}(s)|^2 ds$$

$$\int_{-a}^a 1 \cdot dx = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{4\sin^2 sa}{s^2} ds$$

$$\int_{-\infty}^\infty \frac{\sin^2 sa}{s^2} ds = \frac{2\pi}{4} \cdot 2a = \pi a$$

$$\Rightarrow 2\int_0^\infty \frac{\sin^2 sa}{s^2} ds = \pi a$$

Santashu

R.C.C.

$$\Rightarrow \int_0^\infty \frac{\sin^2 xa}{x^2} dx = \frac{\pi a}{2}$$

Ans