Ex:1 Let F be field of rational numbers. Determine the degree of splitting field of polynomial x^3-2 over F.

Soln Let F be field of rational numbers.

Let $f(x) = x^3-2 \in F[x]$

In the field of complex numbers the three roots of $f(x)$ are $2^{1/3}$, $\omega \cdot 2^{1/3}$, $\omega^2 2^{1/3}$, where $\omega = \frac{-1+\sqrt{3}i}{2}$ and $2^{1/3}$ is real cube root of 2. The polynomial $f(x)$ is irreducible over F. We have degree of $f(x) = 3$. Also $2^{1/3}$ is a root of $f(x)$. Therefore ~~f(x)~~ $2^{1/3}$ is algebraic over F of degree three. Therefore $[F(2^{1/3}):F] = 3$

Let E be splitting field of $f(x)$ over F.

The field $F(2^{1/3})$ cannot split $f(x)$ because subfield of real field cannot contain the complex number $\omega \cdot 2^{1/3}$. Therefore $F(2^{1/3})$ is proper subfield of E.

So we have

$$[E:F] > [F(2^{1/3}):F] = 3.$$

Also $$[E:F] \leq 3! = 6$$

& $$[E:F] = [E:F(2^{1/3})] \cdot [F(2^{1/3}):F]$$

$\Rightarrow$ $[F(2^{1/3}):F]$ is divisor of $[E:F]$

$\Rightarrow$ 3 is divisor of $[E:F]$

Since $[E:F] \leq 6$, $[E:F] > 3$ and 3 is divisor of $[E:F]$

$$\therefore [E:F] = 6$$

Ex 2 Let F be field of rational numbers and $f(x) = x^4 + x^2 + 1$. Show that $F(\omega)$ where $\omega = \frac{-1+\sqrt{3}i}{2}$ is splitting field of $f(x)$. Also determine the degree of the splitting field of $f(x)$ over F.

Sol^n We have

$$x^4 + x^2 + 1 = (x^2+x+1)(x^2-x+1)$$

If in some extension of F, α is a root of x^2+x+1 then $-\alpha$ is root of x^2-x+1 in the same field. Thus if any extension of F splits x^2+x+1, then it will also split x^2-x+1. and consequently it will split x^4+x^2+1.

This splitting field $x^4+x^2+1 \in F[x]$ is the same as that of $x^2+x+1 \in F[x]$

Let $f(x) = x^2+x+1$. In the field of complex numbers the two roots of x^2+x+1 are ω & ω^2.

Since $f(x)$ is irreducible over F and its degree is 2. therefore in any extension of F of degree less than 2, $f(x)$ can not have a root. So if E is splitting field of $f(x)$, then $[E:F] \geq 2$.

Also $[E:F] \leq 2! = 2$

So we must have $[E:F] = 2$

The field $F(\omega)$ contains a root of $f(x)$ i.e. ω.

Since $\omega \in F(\omega)$

$\Rightarrow \omega^2 \in F(\omega)$

Thus $F(\omega)$ contains both the roots ω & ω^2 of $f(x)$. Thus $F(\omega)$ splits x^2+x+1

The polynomial $f(x)$ is irreducible over F. We have $\deg f(x) = 2$ Also ω is a root of $f(x)$. Therefore ω is algebraic over F of degree 2

So we have $[F(\omega):F] = 2$

Hence $F(\omega)$ is splitting field of $f(x)$

Proved