

$$= \frac{M}{\sigma - \sigma_0} \quad \text{if } \sigma > \sigma_0$$

$$\Rightarrow \int_0^{\infty} |e^{-(s-\sigma_0)t} \beta(t) dt| \leq \frac{M}{\sigma - \sigma_0} \quad \text{if } \sigma > \sigma_0$$

$$\Rightarrow \int_0^{\infty} |e^{-(s-\sigma_0)t} \beta(t) dt| \text{ is convergent, if } \sigma > \sigma_0$$

$$\Rightarrow \int_0^{\infty} e^{-(s-\sigma_0)t} \beta(t) dt \text{ converges absolutely, if } \sigma > \sigma_0$$

$$\Rightarrow \int_0^{\infty} e^{-(s-\sigma_0)t} \beta(t) dt \text{ converges for all } s = \sigma + iT \text{ if } \sigma > \sigma_0$$

Hence from eq (1)

$$\int_0^{\infty} e^{-st} d\alpha(t) \text{ converges for all } s = \sigma + iT \text{ for which}$$

$\sigma > \sigma_0$  if it is convergent for  $s_0$  i.e. for  $\sigma = \sigma_0$

Lemma: The region of convergence of  $\int_0^{\infty} e^{-st} d\alpha(t)$  half plane.

Proof: We have proved if  $\int_0^{\infty} e^{-st} d\alpha(t)$  is cgt for  $s = s_0 = \sigma_0 + iT$  i.e.  $\sigma = \sigma_0$  then  $\int_0^{\infty} e^{-st}$  is uniform and absolutely cgt i.e. cgt for  $\sigma > \sigma_0$ .

Now, we shall prove above, if  $\int_0^{\infty} e^{-st} d\alpha(t)$  is dgt. at  $s = s_0$  i.e. for  $\sigma = \sigma_0$ , then  $\int_0^{\infty} e^{-st} d\alpha(t)$  is dgt if  $\sigma > \sigma_0$ .

$$\therefore \int_0^{\infty} e^{-st} d\alpha(t) \text{ is dgt for } s = s_0$$

$$\Rightarrow \int_0^{\infty} e^{-s_0 t} d\alpha(t) \text{ is dgt}$$

$$\Rightarrow \left| \int_0^u e^{-s_0 t} d\alpha(t) \right| > L, \text{ where } L \text{ is any finite num. however large for some } u \text{ in } 0 < u < \infty$$

Take  $\beta(u) = \int_0^u e^{-\sigma_0 t} d\alpha(t)$ , so that  $|\beta(u)| \leq L$  for at least one  $u$  in  $0 \leq u < \infty$ .

Now, as before,

$$\begin{aligned} \int_0^\infty e^{-\sigma t} d\alpha(t) &= \int_0^\infty e^{-(\sigma-\sigma_0)t} d\beta(t) \\ &= \left[ e^{-(\sigma-\sigma_0)t} \beta(t) \right]_0^\infty + (\sigma-\sigma_0) \int_0^\infty e^{-(\sigma-\sigma_0)t} \beta(t) dt \\ &= \lim_{t \rightarrow \infty} e^{-(\sigma-\sigma_0)t} \beta(t) - \beta(0) + (\sigma-\sigma_0) \int_0^\infty e^{-(\sigma-\sigma_0)t} \beta(t) dt \quad (5) \end{aligned}$$

Now,  $|e^{-(\sigma-\sigma_0)t} \beta(t)| = e^{-(\sigma-\sigma_0)t} |\beta(t)| \geq L e^{-(\sigma-\sigma_0)t}$  for at least one  $t$  in  $0 \leq t < \infty$ .

$$\rightarrow \lim_{t \rightarrow \infty} e^{-(\sigma-\sigma_0)t} \beta(t) \geq L \cdot \lim_{t \rightarrow \infty} e^{-(\sigma-\sigma_0)t}$$

$$\rightarrow \lim_{t \rightarrow \infty} e^{-(\sigma-\sigma_0)t} \beta(t) \geq \infty, \text{ if } \sigma < \sigma_0$$

$$\lim_{t \rightarrow \infty} e^{-(\sigma-\sigma_0)t} \beta(t) = \infty, \text{ if } \sigma < \sigma_0$$

From (5), whatever be the nature of  $\sigma - \sigma_0$ ,  $\int_0^\infty e^{-(\sigma-\sigma_0)t} \beta(t) dt$  at

we can say now that

$\int_0^\infty e^{-\sigma t} d\alpha(t)$  is divergent, if  $\sigma < \sigma_0$ .

Thus we have proved that if  $\int_0^\infty e^{-\sigma t} d\alpha(t)$  converges for  $\sigma = \sigma_0$ , the same converges for  $\sigma > \sigma_0$ .

and if  $\int_0^\infty e^{-\sigma t} d\alpha(t)$  diverges for  $\sigma = \sigma_0$ , the same is divergent for  $\sigma < \sigma_0$ .

The line  $\sigma = \sigma_0$  is the axis of convergence.  $\sigma_0$  is the abscissa of convergence and we usually denote  $\sigma_c$  as the abscissa of convergence.

and line  $\sigma = \sigma_c$  as the axis of convergence

We refer the region  $\sigma > \sigma_c$  as the half plane for the region of convergence

$\therefore$  Region of convergence of  $\int_0^\infty e^{-st} dt$  is a half plane

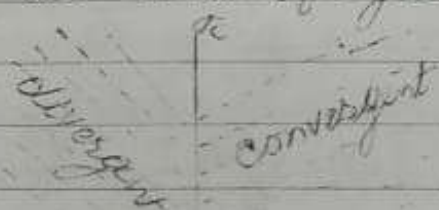
Note From above there are three possibilities arise

- (1) The integral  $\int_0^\infty e^{-st} dt$  converges for no point
- (2) " " " " " " every point
- (3) " " " " " "  $\sigma_c$  and diverges for  $\sigma < \sigma_c$

In case (3), we define  $\sigma_c$  as the abscissa of convergence. the line  $\sigma = \sigma_c$  as the axis of convergence

In case (1)  $\sigma_c = +\infty$

In case (2)  $\sigma_c = -\infty$



case (1)  $\rightarrow +\infty$

$-\infty \leftarrow$  case (2)

We refer the region  $\sigma > \sigma_c$  as the half plane for the region of convergence

x: For  $\sigma_c = 0$  for  $\int_0^\infty e^{-st} dt$

$$\int_0^\infty e^{-st} dt = \left[ \frac{e^{-st}}{-s} \right]_0^\infty = \lim_{t \rightarrow \infty} \frac{e^{-st}}{-s} + \frac{1}{s}$$

Now,  $\left| \frac{e^{-st}}{-s} \right| = \frac{|e^{-st}|}{|s|} = \frac{e^{-\sigma t}}{\sqrt{\sigma^2 + \omega^2}}$ , where  $s = \sigma + i\omega$

$$\lim_{t \rightarrow \infty} \frac{e^{-st}}{-s} = \lim_{t \rightarrow \infty} \frac{e^{-\sigma t}}{\sqrt{\sigma^2 + \omega^2}} = \begin{cases} 0 & \text{if } \sigma > 0 \\ \infty & \text{if } \sigma < 0 \end{cases}$$

$$\Rightarrow \lim_{t \rightarrow \infty} \frac{e^{-st}}{-s} = \begin{cases} 0 & \text{if } \sigma > 0 \\ \infty & \text{if } \sigma < 0 \end{cases}$$

$\therefore \int_0^{\infty} e^{-st} dt$  is convergent for  $\sigma > 0$

and divergent for  $\sigma < 0$   
 $\therefore \sigma = 0$  is the axis of convergence of  $\int_0^{\infty} e^{-st} dt$

cc.  $\sigma_c = 0$  for the same integral

Q.17 Define the Laplace - Stieltjes transformation of a complex function  $x(t)$  of the real variable  $t$  and obtain its region of convergence. R.V. 12

Ans Give the definition of L-S transformation  
 2nd part

Let the integral  $\int_0^{\infty} e^{st} dx(t)$  is convergent  
 $s = s_0 = \sigma_0 + i\tau_0$

$\Rightarrow \int_0^{\infty} e^{s_0 t} dx(t)$  is convergent

So the Corollary 3 and 4 of Theorem 1

a region of convergence of  $\int_0^{\infty} e^{st} dx(t)$  is a half plane

Capital Order of a function  $g(x)$ .

The capital order of a function  $g(x)$  is denoted by  $O(g(x))$

$$f(x) = O(g(x)) \text{ as } x \rightarrow c$$

$\Leftrightarrow \exists$  a +ve constant  $M$  such that

$$|f(x)| \leq M|g(x)|, \quad 0 \leq x \leq c$$

$$f(x) = O(g(x)) \text{ as } x \rightarrow \infty$$

$$\Leftrightarrow \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = k \neq 0$$

clearly if  $g(x) = 1$

$$f(x) = O(g(x)) \Leftrightarrow f(x) = O(1)$$

$$\Leftrightarrow |f(x)| \leq M \quad \text{ie. } |f(x)| \leq M$$

If  $g(x) = e^{rt}$ ,  $r$  is real number

$$f(x) = O(g(x)) \Leftrightarrow f(x) = O(e^{rt})$$

$$\Leftrightarrow |f(x)| \leq M |e^{rt}|$$

$$\Rightarrow |f(x)| \leq M e^{rt} \quad \because |e^{rt}| = e^{rt}$$

Small order of a function  $g(x)$ :-

Small order of a function is denoted by  $O(g(x))$

$$f(x) = O(g(x)) \text{ as } x \rightarrow c$$

$$\Leftrightarrow \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 0 \text{ (zero)}$$

$$\Leftrightarrow \left| \frac{f(x)}{g(x)} - 0 \right| < \epsilon, \quad 0 < x < c$$

$$\Leftrightarrow \left| \frac{f(x)}{g(x)} \right| < \epsilon$$

$$\Leftrightarrow |f(x)| < \epsilon |g(x)|, \quad \forall \epsilon > 0$$

$$\Leftrightarrow |f(x)| \leq M |g(x)|; \text{ where } M > \epsilon$$

If  $g(x) = 1$ ,  $f(x) = O(g(x))$   
ie  $f(x) = O(1)$

$$\Rightarrow |f(x)| < \epsilon, \quad \forall \epsilon > 0$$

2.1) If  $\alpha(t) = O(e^{rt})$  as  $t \rightarrow \infty$  for some real  $r$ , then the integral  $\int_0^{\infty} e^{-st} d\alpha(t)$  converges for

R.U. -

$$\because \alpha(t) = O(e^{rt}), \quad t \rightarrow \infty$$

$\Rightarrow \exists$  a +ve constant  $M$  such that  
 $|\alpha(t)| \leq M(e^{rt})$  — ①

$$\text{Now, } \int_0^{\infty} e^{-st} d\alpha(t) = \left[ e^{-st} \alpha(t) \right]_0^{\infty} + s \int_0^{\infty} e^{-st} \alpha(t) dt$$

(Integrating by part) (Int. by part)

(33)

$$= \lim_{t \rightarrow \infty} [e^{-st} \alpha(t)] - 0 \cdot 0 + s \int_0^{\infty} e^{-st} \alpha(t) dt = 0$$

Now,  $|e^{-st} \alpha(t)| = e^{-\sigma t} |\alpha(t)| \leq e^{-\sigma t} M e^{\gamma t}$  (by eq 1)  
 $= M e^{(\gamma - \sigma)t}$  1.3

$$\therefore \lim_{t \rightarrow \infty} e^{-st} \alpha(t) \leq M \lim_{t \rightarrow \infty} e^{(\gamma - \sigma)t}$$

$$= 0 \quad \text{if } \sigma > \gamma$$

$$\therefore \lim_{t \rightarrow \infty} e^{-st} \alpha(t) \leq 0 \quad \text{if } \sigma > \gamma$$

$$\lim_{t \rightarrow \infty} e^{-st} \alpha(t) = 0 \quad \text{if } \sigma > \gamma \quad (\because 1/x \rightarrow 0 \text{ as } x \rightarrow \infty)$$

$$\lim_{t \rightarrow \infty} e^{-st} \alpha(t) = 0$$

Again,  $\int_0^{\infty} |e^{-st} \alpha(t)| dt \leq M \int_0^{\infty} e^{-(\sigma - \gamma)t} dt$  (by eqs)  
 $= M \left[ \frac{e^{-(\sigma - \gamma)t}}{-(\sigma - \gamma)} \right]_0^{\infty} = \frac{M}{\sigma - \gamma} \quad \text{if } \sigma > \gamma$

$$\therefore \int_0^{\infty} e^{-st} \alpha(t) dt \text{ is absolutely cgt. if } \sigma > \gamma$$

$$\int_0^{\infty} e^{-st} \alpha(t) dt \text{ is also cgt. if } \sigma > \gamma$$

$\therefore$  from eq (2), we have

$$\int_0^{\infty} e^{-st} d\alpha(t) = 0 - \alpha(0) + (\text{a convergent integral})$$

if  $\sigma > \gamma$

$$\therefore \int_0^{\infty} e^{-st} d\alpha(t) = \text{a convergent integral if } \sigma > \gamma$$

$\because \alpha(t)$  being order of a function  $e^{\gamma t}$ , it must be bounded i.e.  $\alpha(t)$  is finite  $\forall t$ , so  $\alpha(0)$  is finite

$$\therefore \int_0^{\infty} e^{-st} d\alpha(t) \text{ is cgt if } \sigma > \gamma$$