## Second order P.D.E :

$r = \frac{\partial^2 z}{\partial x^2}$ $\quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ $\quad t = \frac{\partial^2 z}{\partial y^2}$

ie $r = \frac{\partial p}{\partial x}$ $\quad s = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ $\quad t = \frac{\partial q}{\partial y}$

**Ex 1** Solve $xys = 1$.

$$\Rightarrow s = \frac{1}{xy}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{xy}$$

Integrating w.r.to $x$

$$\frac{\partial z}{\partial y} = \frac{1}{y}\log x + f(y)$$

Integrating w.r.to $y$

$$z = \log y \cdot \log x + \int f(y)\,dy + F(x)$$

$$\Rightarrow z = \log x \cdot \log y + \phi(y)\,dy + F(x)$$

Ans

**Ex 2** Solve $xr + p = 9x^2y^3$.

Sol^n Given $xr + p = 9x^2y^3$

$$\Rightarrow r + \frac{p}{x} = 9xy^3$$

$$\Rightarrow \frac{\partial p}{\partial x} + \frac{p}{x} = 9xy^3$$

Linear form of differential eqn $\left(\frac{dy}{dx} + Py = Q\right)$

Integrating factor $IF = e^{\int \frac{1}{x} dx}$

$$= e^{\log x}$$

$$= x.$$

Solⁿ $p(x) = \int 9xy^3 \cdot x \, dx + F(y)$

$$\Rightarrow px = 9\frac{x^3}{3}y^3 + F(y)$$

$$\Rightarrow px = 3x^3y^3 + F(y)$$

$$\Rightarrow p = 3x^2y^3 + \frac{1}{x}F(y)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 3x^2y^3 + \frac{1}{x}F(y)$$

Integrating w.r.to $x$

$$z = 3\frac{x^3}{3} \cdot y^3 + \log x \cdot F(y) + G(y).$$

$$\Rightarrow z = \frac{1}{3}x^3y^3 + \log x \cdot F(y) + G(y)$$

Ans

Gen. solⁿ:

**Ex (3)** Solve $s = 2x + 2y$

Solⁿ Given $s = 2x + 2y$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = 2x + 2y$$

Integrating w.r.to $x$

$$\frac{\partial z}{\partial y} = x^2 + 2xy + f(y)$$

Now integrating w.r.to $y$

$$z = x^2y + \cancel{xy}^2 + \int f(y)\,dy + F(x)$$

$$\Rightarrow z = x^2y + xy^2 + G(y) + F(x)$$

When $G(y) = \int f(y)\,dy$

Ex(4) $t - xq = x^2$

Sol$^n$ Given $t - xq = x^2$

$$\Rightarrow \frac{\partial q}{\partial y} - qx = x^2 \quad \text{(It is Linear diff eq}^n\text{)}$$

$$I.F = e^{\int -x \, dy}$$
$$= e^{-xy}$$

Sol$^n$. $$q \cdot (e^{-xy}) = \int x^2 \cdot e^{-xy} dy + F(x)$$

$$\Rightarrow q \cdot e^{-xy} = x^2 \cdot \frac{e^{-xy}}{-x} + F(x)$$

$$\Rightarrow q e^{-xy} = -x e^{-xy} + F(x)$$

$$\Rightarrow q = -x + e^{xy} F(x)$$

$$\Rightarrow \frac{\partial z}{\partial y} = -x + e^{xy} F(x)$$

Integrating w.r.to $y$

$$z = -xy + \frac{e^{xy}}{x} \cdot F(x) + G(x)$$

$$\Rightarrow z = -xy + e^{xy} \phi(x) + G(x)$$

where $\phi = \frac{F(x)}{x}$ Ans

Gen. sol$^n$

## Monge's Method for $Rr + Ss + Tt = V$.

Given $Rr + Ss + Tt = V$ —— (1) where $R, S, T, V$ are functions of $x, y, z, p$ & $q$.

We know that

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = r\,dx + s\,dy$$

Similarly $$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = s\,dx + t\,dy$$

$$\Rightarrow r = \frac{dp - s\,dy}{dx} \quad \& \quad t = \frac{dq - s\,dx}{dy}$$

Putting in eqn (1)

$$R\left(\frac{dp - s\,dy}{dx}\right) + Ss + T\left(\frac{dq - s\,dx}{dy}\right) = V.$$

$$\Rightarrow \frac{R(dp - s\,dy)dy + Ss\,dx\,dy + T(dq - s\,dx)dx}{dx\,dy} = V$$

$$\Rightarrow R\,dp\,dy - Rs\,dy^2 + Ss\,dx\,dy + T\,dq\,dx + Ts\,dx^2 = V\,dx\,dy$$

$$\Rightarrow (R\,dp\,dy - V\,dx\,dy + T\,dq\,dx) - s(R\,dy^2 - S\,dx\,dy + T\,dx^2) = 0 \quad \text{—— (2)}$$

If some relation between $x, y, z, p$ & $q$ makes each of above brackets zero, the relation will satisfy eqn (2)

$$\Rightarrow R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0 \quad \text{—— (3)}$$

$$\& \quad R\,dp\,dy - V\,dx\,dy + T\,dq\,dx = 0 \quad \text{—— (4)}.$$

Eqn (3) & (4) are called Monge's subsidiary eqn.

If Eqn (3) is resolved into two linear eqns in $dx$ & $dy$ such that

$$dy - m_1 dx = 0 \quad \text{——} \quad (5)$$

$$dy - m_2 dx = 0 \quad \text{——} \quad (6)$$

Taking eqn (4) & (5) if necessary also take

$$dz = p\,dx + q\,dy$$

and obtain two integrals

$$u_1 = a \quad \& \quad v_1 = b$$

Then the relation

$$u_1 = f_1(v_1) \quad \text{——} \quad (7)$$

is intermediate integral.

Similarly Taking eqn (4), (6) if necessary also take $dz = p\,dx + q\,dy$ and obtain two integrals

$$u_2 = a' \quad \& \quad v_2 = b'$$

Then the relation

$$u_2 = f_2(v_2) \quad \text{——} \quad (8)$$

is intermediate integral

Now from (7) & (8) find the value of $p$ & $q$ and put in $dz = p\,dx + q\,dy$ and integrate it to get complete integral.