## FOURIER TRANSFORM (16)

**Problem (11)** Find finite Fourier Sine and Cosine transform of $f(x) = x^2,\ 0 < x < 4$.

**Answer (i)**

$$F_S\{f(x)\} = \int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$$

$$\Rightarrow F_S\{f(x)\} = \int_0^4 f(x)\sin\frac{n\pi x}{4}\,dx = \int_0^4 x^2\sin\frac{n\pi x}{4}\,dx$$

$$\bar{f}_S(n) = \left[x^2\,\frac{-\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right]_{x=0}^{4} - \int_0^4 2x\cdot\frac{-\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}\,dx$$

$$= -\frac{4}{n\pi}\left[-0 + 4^2\cos n\pi\right] + \frac{8}{n\pi}\int_0^4 x\cos\frac{n\pi x}{4}\,dx$$

$$= -\frac{4^3\cos n\pi}{n\pi} + \frac{8}{n\pi}\left[x\cdot\frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}} - \int 1\cdot\frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right]_{x=0}^{4}$$

$$= -\frac{4^3\cos n\pi}{n\pi} + \frac{8}{n\pi}\left[\frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{4}{n\pi}\cdot\frac{\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right]_{x=0}^{4}$$

$$= -\frac{4^3\cos n\pi}{n\pi} + \frac{8}{n\pi}\left[\frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_{x=0}^{4}$$

$$= -\frac{64}{n\pi}\cos n\pi + \frac{8}{n\pi}\left[\frac{16}{n\pi}\sin n\pi + \frac{16}{n^2\pi^2}\cos n\pi - \left(0 + \frac{16}{n^2\pi^2}\cdot 1\right)\right]$$

$$= -\frac{64}{n\pi}\cos n\pi + \frac{8}{n\pi}\left[0 + \frac{16}{n^2\pi^2}(\cos n\pi - 1)\right]$$

$$= -\frac{64}{n\pi}\cos n\pi + \frac{128}{n^3\pi^3}(\cos n\pi - 1) \quad \text{Ans.}$$

(ii) $F_c\{f(x)\} = \bar{f}_c(n) = \int_0^4 f(x)\cos\frac{n\pi x}{4}\,dx$

$$= \int_0^4 x^2 \cos\frac{n\pi x}{4}\,dx$$

$$= \left[x^2 \frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right]_{x=0}^4 - \int_0^4 2x \cdot \frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}}\,dx$$

$$= \left[\frac{4x^2}{n\pi}\sin\frac{n\pi x}{4}\right]_{x=0}^4 - \int_0^4 \frac{8x}{n\pi}\sin\frac{n\pi x}{4}\,dx$$

$$= \left(\frac{64}{n\pi}\sin n\pi - 0\right) - \frac{8}{n\pi}\left[x\cdot\frac{-\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}} - \int 1\cdot\frac{-\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}\,dx\right]_{x=0}^4$$

$$= \frac{64}{n\pi}\cdot 0 + \frac{8}{n\pi}\cdot\frac{4}{n\pi}\left[x\cos\frac{n\pi x}{4}\right]_0^4 - \frac{8}{n\pi}\cdot\frac{4}{n\pi}\left[\frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right]_0^4$$

$$= \frac{32}{n^2\pi^2}\left[4\cos n\pi - 0\cos 0\right] - \frac{128}{n^3\pi^3}\left[\sin n\pi - \sin 0\right]$$

$$= \frac{128}{n^2\pi^2}\cos n\pi \quad \text{Ans.}$$

Problem (12) find the finite sine transform of
(i) $e^{ax}$ (ii) $\sin ax$ (iii) $\cos ax$ (iv) $\frac{x^3}{\pi}$

Answer $\bar{f}_s(s) = \int_0^\pi f(x)\sin\left(\frac{s\pi x}{\pi}\right)dx = \int_0^\pi f(x)\sin(sx)\,dx$

(i) $F_s\{e^{ax}\} = \int_0^\pi e^{ax}\sin sx\,dx$

$$= \left[\frac{e^{ax}(a\sin sx - s\cos sx)}{a^2+s^2}\right]_{x=0}^\pi$$

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$$F_s\{e^{ax}\} = \frac{1}{a^2+s^2}\left[e^{ax}(a\sin sx - s\cos sx)\right]_{x=0}^{\pi}$$

$$= \frac{1}{a^2+s^2}\left[e^{a\pi}(a\sin s\pi - s\cos s\pi) - e^0(a\sin 0 - s\cos 0)\right]$$

$$= \frac{1}{a^2+s^2}\left[e^{a\pi}(0 - s\cos s\pi) + s.1\right]$$

$$= \frac{1}{a^2+s^2}\left[s - s.e^{a\pi}\cos s\pi\right]$$

$$= \frac{s}{a^2+s^2}\left[1 - e^{a\pi}.(-1)^s\right] \quad \underline{\text{Ans.}}$$

(ii) $F_s\{\sin ax\} = \int_0^{\pi} \sin ax . \sin xs \, dx$

$$= \frac{1}{2}\int_0^{\pi} 2\sin ax \sin sx \, dx$$

$$= \frac{1}{2}\int_0^{\pi} \{\cos(a-s)x - \cos(a+s)x\} \, dx$$

$$= \frac{1}{2}\left[\frac{\sin(a-s)x}{a-s} - \frac{\sin(a+s)x}{a+s}\right]_{x=0}^{\pi}$$

$= \frac{1}{2}(0-0) = 0$, if $a \neq s$ and $a, s$ are integers.

Also if $s = a$, then

$$F_s\{\sin ax\} = \frac{1}{2}\int_0^{\pi} 2\sin^2 sx \, dx = \frac{1}{2}\int_0^{\pi}(1-\cos 2sx)\,dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2sx}{2s}\right]_0^{\pi} = \frac{\pi}{2}$$

$$\therefore\ F_s\{\sin ax\} = \begin{cases} 0, & \text{if } a \neq s \text{ and } a, s \text{ are integers} \\ \frac{\pi}{2}, & \text{if } a = s. \end{cases}$$

Ans

(iii) $F_s\{\cos ax\} = \int_0^{\pi} \cos ax \sin xs \, dx$

$$= \frac{1}{2}\int_0^{\pi} 2 \sin sx \cos ax \, dx$$

$$= \frac{1}{2}\int_0^{\pi} [\sin(s+a)x + \sin(s-a)x] \, dx$$

$$= \frac{1}{2}\left[-\frac{\cos(s+a)x}{s+a} - \frac{\cos(s-a)x}{s-a}\right]_0^{\pi}$$

$$= -\frac{1}{2(s+a)}\left[\cos(s+a)x + \cos(s-a)x\right]_{x=0}^{\pi}$$

$$= -\frac{1}{2(s+a)}\left[\cos(s+a)\pi + \cos(s-a)\pi - (\cos 0 + \cos 0)\right]$$

$$= -\frac{1}{2(s+a)}\left[\cos(s\pi + a\pi) + \cos(s\pi - a\pi) - 2\right]$$

$$= -\frac{1}{2(s+a)}\left[2\cos s\pi \cos a\pi - 2\right]$$

$$= -\frac{1}{s+a}\left[(-1)^s \cos a\pi - 1\right]$$

$$= \frac{1}{s+a}\left[1 - (-1)^s \cos a\pi\right], \text{ if } s \text{ is } +\text{ve}.$$

Ans

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(iv) $F_s(x^3) = \int_0^{\pi} x^3 \sin xs\, dx$

$$= \left(-\frac{x^3 \cos xs}{s}\right)_0^{\pi} - \int_0^{\pi} 3x^2 \cdot \frac{-\cos xs}{s}\, dx$$

$$= \left(\frac{-\pi^3}{s}\cos \pi s - 0\right) + \frac{3}{s}\int_0^{\pi} x^2 \cos xs\, dx$$

$$= -\frac{\pi^3}{s}\cos \pi s + \frac{3}{s}\left[x^2 \cdot \frac{\sin xs}{s} - \int 2x \cdot \frac{\sin xs}{s}\, dx\right]_0^{\pi}$$

$$= -\frac{\pi^3}{s}\cos \pi s + \frac{3}{s^2}\left[x^2 \sin xs\right]_0^{\pi} - \frac{6}{s^2}\int_0^{\pi} x \sin xs\, dx$$

$$= -\frac{\pi^3}{s}\cos \pi s + \frac{3}{s^2}(0-0) - \frac{6}{s^2}\left[x\left(\frac{-\cos xs}{s}\right) - \int 1 \cdot \frac{-\cos xs}{s}\, dx\right]_0^{\pi}$$

$$= -\frac{\pi^3}{s}\cos \pi s + \frac{6}{s^3}\left[x \cos xs\right]_0^{\pi} - \frac{6}{s^3}\left(\frac{\sin xs}{s}\right)_{x=0}^{\pi}$$

$$= -\frac{\pi^3}{s}\cos \pi s + \frac{6}{s^3}\left[\pi \cos \pi s - 0\right] - \frac{6}{s^4}\left[0-0\right]$$

$$= -\frac{\pi^3}{s}\cos \pi s + \frac{6\pi}{s^3}\cos \pi s$$

$$= \frac{\pi}{s}\cos \pi s\left(\frac{6}{s^2} - \pi^2\right)$$

$$= \frac{\pi}{s}(-1)^s\left\{\frac{6}{s^2} - \pi^2\right\}.$$

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