Ex Solve $r = a^2 t$.

Ans: We know that

$$dp = r\,dx + s\,dy$$

and $$dq = s\,dx + t\,dy$$

$$\Rightarrow r = \frac{dp - s\,dy}{dx} \quad \& \quad t = \frac{dq - s\,dx}{dy}$$

Putting in given differential eqn we get

$$\frac{dp - s\,dy}{dx} = a^2 \frac{dq - s\,dx}{dy}$$

$$\Rightarrow dp\,dy - s\,dy^2 = a^2 dq\,dx - a^2 s\,dx^2$$

$$\Rightarrow (dp\,dy - a^2 dq\,dx) - s(dy^2 - a^2 dx^2) = 0$$

Monge's subsidiary equations are

$$dp\,dy - a^2 dq\,dx = 0 \quad \text{——— (1)}$$

$$dy^2 - a^2 dx^2 = 0 \quad \text{——— (2)}$$

From (2) we get

$$(dy - a\,dx)(dy + a\,dx) = 0$$

$$\Rightarrow dy - a\,dx = 0 \quad \text{——— (3)}$$

$$\& \quad dy + a\,dx = 0 \quad \text{——— (4)}$$

From (1) & (3)

$$dp \cdot a\,dx - a^2 dq\,dx = 0$$

$$\Rightarrow dp - a\,dq = 0 \quad \text{——— (5)}$$

From (3) & (5) $y - ax = a_1$

$$\& \quad p - aq = b_1$$

$$\Rightarrow \quad p - aq = f_1(y - ax) \quad \text{Intermediate integral} \quad \text{(6)}$$

Similarly from (4) & (1)

$$-dp.\,adx - a^2 dq\, dx = 0$$

$$\Rightarrow \quad dp + adq = 0 \quad \text{(7)}$$

From (4) & (7) $\quad y + ax = a_2$

$$p + aq = b_2$$

$$\Rightarrow \quad p + aq = f_2(y + ax) \quad \text{Intermedial integral.} \quad \text{(8)}$$

From (6) & (8)

$$p = \frac{1}{2}\left\{f_1(y - ax) + f_2(y + ax)\right\}$$

$$q = \frac{1}{2a}\left\{f_2(y + ax) - f_1(y - ax)\right\}$$

Putting in $dz = p\,dx + q\,dy$

$$dz = \frac{1}{2}\left\{f_1(y - ax) + f_2(y + ax)\right\}dx + \frac{1}{2a}\left\{f_2(y + ax) - f_1(y - ax)\right\}dy$$

$$\Rightarrow dz = \frac{1}{2a} f_1(y - ax)(adx - dy) + \frac{1}{2a} f_2(y + ax)(adx + dy)$$

$$\Rightarrow 2a\,dz = f_2(y + ax).(dy + adx) - f_1(y - ax)(dy - adx)$$

On integrating.

$$2az = \phi_2(y + ax) + \phi_1(y - ax) \quad \text{Ans}$$

$$\left\{ \begin{array}{l} \because d(y - ax) = dy - adx \\ d(y + ax) = dy + adx \end{array} \right.$$

Solve $2x^2r - 5xys + 2y^2t + 2px + 2qy = 0$.

Sol^n We know that

$$r = \frac{dp - s\,dy}{dx} \quad \& \quad t = \frac{dq - s\,dx}{dy}$$

Putting in given partial differential equation.

$$2x^2\left(\frac{dp - s\,dy}{dx}\right) - 5xys + 2y^2\left(\frac{dq - s\,dx}{dy}\right) + 2px + 2qy = 0$$

$$\Rightarrow \left\{2x^2 dp\,dy + 2(px+qy)\,dx\,dy + 2y^2 dq\,dx\right\}$$
$$- s\left\{2x^2 dy^2 + 5xy\,dx\,dy + 2y^2 dx^2\right\} = 0$$

Monge's subsidiary equations are

$$2x^2 dp\,dy + 2(px+qy)\,dx\,dy + 2y^2 dq\,dx = 0 \quad \text{—— (1)}$$
$$2x^2 dy^2 + 5xy\,dx\,dy + 2y^2 dx^2 = 0 \quad \text{—— (2)}$$

Solving (2)

$$(2x\,dy + y\,dx)(x\,dy + 2y\,dx) = 0$$

$$\Rightarrow 2x\,dy + y\,dx = 0 \quad \text{—— (3)}$$
$$x\,dy + 2y\,dx = 0 \quad \text{—— (4)}$$

From (1) & (3)

$$\cancel{2x^2 dp\left(-\frac{y}{2x}dx\right) + 2(px+qy)\,dx\left(\frac{-y}{2x}dx\right) + 2y^2 dx^2 = 0}$$

$$2x^2 dp\cdot\left(-\frac{y}{2x}dx\right) + 2px\,dx\cdot\left(-\frac{y}{2x}dx\right) + 2qy\,dx\,dy + 2y^2 dq\,dx = 0$$

$$\Rightarrow -xy\,dp\cdot dx - py\,dx^2 + 2qy\,dx\,dy + 2y^2 dq\,dx = 0$$

Dividing by $-y\,dx$

$$x\,dp + p\,dx - 2q\,dy - 2y\,dq = 0 \quad \text{--- (5)}$$

Solving (5) & (3)

$$(3) \Rightarrow \frac{2\,dy}{y} + \frac{dx}{x} = 0$$

$$\Rightarrow \log y^2 + \log x = \log A$$

$$\Rightarrow \log xy^2 = \log A$$

$$xy^2 = A \quad \text{--- (6)}$$

$$(5)\quad (x\,dp + p\,dx) - 2(q\,dy + y\,dq) = 0$$

$$\Rightarrow d(xp) - 2\,d(yq) = 0$$

$$\Rightarrow xp - 2yq = B \quad \text{--- (7)}$$

From (6) & (7)

$$xp - 2yq = f_1(xy^2) \quad \text{--- (8)}$$ — intermediate integr

Similarly taking (1) & (4) & finding intermediate

$$2px - yq = f_2(x^2y) \quad \text{--- (9)}$$ — intermediate integr

Solving (8) & (9) we get

$$p = \frac{1}{3x}\left\{2f_2(x^2y) - f_1(xy^2)\right\}$$

$$q = \frac{1}{3y}\left\{f_2(x^2y) - 2f_1(xy^2)\right\}$$

Putting in $dz = p\,dx + q\,dy$ we get

$$dz = \frac{1}{3x}\left\{2f_2(x^2y) - f_1(xy^2)\right\}dx + \frac{1}{3y}\left\{f_2(x^2y) - 2f_1(xy^2)\right\}dy$$

$$\Rightarrow dz = f_2(x^2y)\left(\frac{2dx}{3x} + \frac{dy}{3y}\right) - f_1(xy^2)\left(\frac{dx}{3x} + \frac{2dy}{3y}\right)$$

$$\Rightarrow dz = \frac{1}{3}\left[f_2(x^2y)\left(\frac{2dx}{x} + \frac{dy}{y}\right) - f_1(xy^2)\left(\frac{dx}{x} + \frac{2dy}{y}\right)\right]$$

$$\Rightarrow dz = \frac{1}{3}\left[f_2(x^2y)\, d\left(\log(x^2y)\right) - f_1(xy^2)\, d\left(\log(xy^2)\right)\right]$$

On integrating

$$z = \frac{1}{3}\left[\phi_2(x^2y) + \phi_1(xy^2)\right]$$

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