Q. If $\alpha(t) = O(e^{\gamma t})$, $t \to \infty$, for some real γ, then $\int_0^\infty e^{-st} d\alpha(t)$ converges absolutely for $\sigma > \gamma$ [R.U-7.]

Ans

$\because \alpha(t) = O(e^{\gamma t})$, $t \to \infty$

$\Rightarrow \exists$ a +ve constant M such that

$|\alpha(t)| \leq M e^{\gamma t}$

$\Rightarrow \alpha(t) \leq |\alpha(t)| \leq M e^{\gamma t}$

$\Rightarrow d\alpha(t) \leq M \gamma e^{\gamma t} \, dt$

Now, $\int_0^\infty |e^{-st} d\alpha(t)| \leq \int_0^\infty |e^{-st} M \gamma e^{\gamma t} dt|$

$= |M\gamma| \int_0^\infty e^{-\sigma t} e^{\gamma t} dt$

$= |M\gamma| \int_0^\infty e^{-(\sigma-\gamma)t} dt$

$\{\because s = \sigma + i\tau \Rightarrow |e^{-st}| = e^{-\sigma t}$ and $|e^{\gamma t}| = e^{\gamma t}$

$= M|\gamma| \cdot \left[\frac{e^{-(\sigma-\gamma)t}}{-(\sigma-\gamma)} \right]_0^\infty$

$= \frac{M|\gamma|}{(\sigma-\gamma)}$; if $\sigma > \gamma$

$\Rightarrow \int_0^\infty |e^{-st} d\alpha(t)| \leq \frac{M|\gamma|}{(\sigma-\gamma)}$ if $\sigma > \gamma$

$\Rightarrow \int_0^\infty |e^{-st} d\alpha(t)|$ is convergent if $\sigma > \gamma$

$\Rightarrow \int_0^\infty e^{-st} d\alpha(t)$ is absolutely convergent if $\sigma >$

Note The theorem ⟨2.1⟩ can be proved by the me of before Q.

$\therefore \int_0^\alpha e^{-st} d\alpha(t)$ is absolutely cgt. i

> Define <i> the generating Determining function
<ii> the generating function . of the Laplace Transform [R.U.-72]

Ans: Laplace Transform means <1> Laplace stieltje's integral denoted by $\int_0^\infty e^{-st}\, d\alpha(t)$

<2> Laplace integral denoted by
$$\int_0^\infty e^{-st} \varphi(t)\, dt$$

When $\int_0^\infty e^{-st} d\alpha(t)$ is convergent for all s. we denote by $f(s)$ i.e. $f(s) = \int_0^\infty e^{-st} d\alpha(t)$ and $f(s)$ is called generating function and $\alpha(t)$ is called Determining function.

Next if $\int_0^\infty e^{-st} \varphi(t)\, dt$ converges to function say $F(s)$

$$\text{i.e. } F(s) = \int_0^\infty e^{-st} \varphi(t)\, dt$$

then $F(s)$ is called generating function and φ is called Determining function

Analytic characteristic of the Generating Function $f(s)$: If the integral $f(s) = \int_0^\infty e^{-st} d\alpha(t)$ c... for $\sigma > \sigma_c < \infty$, then $f(s)$ is analytic for $\sigma > \sigma$...

$$f^k(s) = \int_0^\infty e^{-st} (-t)^k d\alpha(t)$$

If $f(s)$ be the generating function of La... stieltje's integral: which is convergent for ... then $f(s)$ is analytic for $\sigma > \sigma_c$ and

$$f^k(s) = \int^\infty e^{-st} (-t)^k d\alpha(t)$$

heorem (6.1) If $\alpha(t)$ is a normalised function of b.v. in $(0,1)$ such that $\int_0^1 t^n d\alpha(t) = 0$, $n = 0,1,2,3,\dots$ then $\alpha(t)$ is identically zero in $0 \le t \le 1$ [R.U-74]

Proof :- Given, $\alpha(t)$ is normalised function in $(0,1)$

$$\Rightarrow \alpha(0) = 0$$

Also, given $\int_0^1 t^n d\alpha(t) = 0$, $n = 0,1,2,3,\dots$

Particularly taking $n = 0$, we have

$$\int_0^1 d\alpha(t) = 0 \Rightarrow [\alpha(t)]_0^1 = 0$$

$$\Rightarrow \alpha(1) - \alpha(0) = 0$$

$$\Rightarrow \alpha(1) = 0 \qquad \{\because \alpha(0) = 0, \text{ by above}\}.$$

$\therefore \alpha(t) = 0$, when $t = 0, 1$

Next, we shall prove that $\alpha(t) = 0$, when $0<t<1$

For $n = n+1$, $\int_0^1 t^{n+1} d\alpha(t) = 0$ also

$$\Rightarrow [t^{n+1}\alpha(t)]_0^1 - (n+1)\int_0^1 t^n \alpha(t)\,dt = 0$$

$$\Rightarrow -(n+1)\int_0^1 t^n \alpha(t)\,dt = 0 \qquad \{\because \alpha(1) = 0, \alpha(0) = 0\}$$

$$\Rightarrow \int_0^1 t^n \alpha(t)\,dt = 0; \quad n = 0,1,2,3,\dots$$

Putting $\int_0^t \alpha(t)\,dt = \beta(t)$

so that $\beta(0) = 0$ & $\beta(1) = \int_0^1 \alpha(t)\,dt = 0$

$$\Rightarrow \alpha(t)\,dt = d\beta(t)$$

$\therefore$ from above, we have

$$\int_0^1 t^n d\beta(t) = 0, \quad n = 0,1,2,3,\dots$$

where $\beta(1) = \beta(0) = 0$

By Weierstrass approximation theorem $\exists$ a poly. $p(t)$ such that

$$|\overline{\beta(t)} - p(t)| < \epsilon, \quad 0 \le t \le 1$$

Taking $p(t) = \sum_{k=0}^{n} a_k t^k$

$$\therefore p(t) \cdot \beta(t) = \sum_{k=0}^{n} a_k t^k \beta(t)$$

$$\Rightarrow \int_0^1 p(t)\beta(t)\,dt = \sum_{k=0}^{n} a_k \int_0^1 t^k \beta(t)\,dt = 0$$

$$\therefore \int_0^1 |\beta(t)|^2\,dt = \int_0^1 \beta(t)\overline{\beta(t)}\,dt$$

$$= \int_0^1 \beta(t)\cdot\overline{\beta(t)}\,dt - \int_0^1 p(t)\cdot\beta(t)\,dt$$

$$= \int_0^1 \beta(t)\left[\overline{\beta(t)} - p(t)\right]dt \quad (\because \text{2nd Integral} = 0)$$

$$\le \int_0^1 |\beta(t)| \cdot |\overline{\beta(t)} - p(t)|\,dt$$

$$< \epsilon \int_0^1 |\beta(t)|\,dt$$

$$\Rightarrow \int_0^1 |\beta(t)|^2\,dt < \epsilon \int_0^1 |\beta(t)|\,dt \quad \forall\, \epsilon > 0$$

As ϵ being arbitrary, this inequality is true $\forall\, \epsilon$. For which we have $\beta(t) = 0$, identically

$$\Rightarrow \int_0^t \alpha(t)\,dt = 0 \quad \forall\, t$$

$$\Rightarrow \alpha(t) = 0 \text{ ; at all points of continuity}$$

Let t be a point of discontinuity, which will be simple kind. Since $\alpha(t)$ is of b.v. so that we have

$\alpha(t+)$, $\alpha(t-)$ both exists

and $\alpha(t) = \dfrac{\alpha(t+) + \alpha(t-)}{2}$ $\{\because \alpha(t)$ is normalised

$= \dfrac{0+0}{2}$	$t+ =$ a point of contin
	$\Rightarrow \alpha(t+) = 0$
$= 0$	lly, $t- =$ a pt. of conti
	$\Rightarrow \alpha(t-) = 0$

$\therefore\ \alpha(t) = 0$ at the point of discontinuity

So $\alpha(t) = 0$, $\forall\ t$, $0 \le t \le 1$

i.e. $\alpha(t) = 0$ identically

orollary 6.1) If $\int_{0+}^{1-} t^n \, d\alpha(t) = 0$, $n = 0, 1, 2,$ & if $\alpha(t)$ is normalised, then $\alpha(t) = 0$, $0 < t < 1$

Proof :-

$$\int_{0+}^{1-} t^n \, d\alpha(t) = 0 \Rightarrow \int_{0+}^{1-} t^n \, d\alpha(t) \text{ exists and } = 0$$

$\Rightarrow \alpha(t)$ is of b.v. in $(0+, 1-)$ i.e. $(\epsilon, 1-\epsilon)$, for $\forall$ +ve ϵ

When $n = 0$, $\int_{0+}^{1-} t^n \, d\alpha(t) = 0$ becomes

$$\int_{0+}^{1-} d\alpha(t) = 0 \Rightarrow [\alpha(t)]_{0+}^{1-} = 0$$

$$\Rightarrow \alpha(1-) - \alpha(0+) = 0$$

$$\Rightarrow \alpha(1-) = \alpha(0+)$$

Again, since $\alpha(t)$ is normalised in $(0^+, 1^-)$

$$\Rightarrow \alpha(0+) = 0$$

So $\alpha(1-) = 0$ [from above]

i.e. $\alpha(t) = 0$, for $t = 0^+, 1^-$

$$\therefore \int_{0+}^{1-} t^n \, d\alpha(t) = 0, \qquad n = 0, 1, 2, 3, -$$

where $\alpha(0+) = 0$, $\alpha(1-) = 0$

$$\Rightarrow \alpha(t) = 0, \quad 0+ \le t \le 1-$$

i.e. $\alpha(t) = 0$, when $0 < t < 1$

ary 6.1 :- If $\varphi(t)$ belongs to $\mathcal{L}[0,1]$ & if $\int_0^1 t^n \varphi(t) \, dt =$ then $\varphi(t) = 0$ almost everywhere $n = 0, 1, 2, 3 - -$

:- If $\varphi(t)$ is a normalised function of b.v. in $\mathcal{L}[0,$