Ex. Solve $t - r\sec^4 y = 2q\tan y$.

Sol^n Given $t - r\sec^4 y = 2q \cdot \tan y$

Putting $r = \dfrac{dp - s\,dy}{dx}$ & $t = \dfrac{dq - s\,dx}{dy}$

$$\left(\frac{dq - s\,dx}{dy}\right) - \left(\frac{dp - s\,dy}{dx}\right)\sec^4 y = 2q \cdot \tan y$$

$$\Rightarrow \left(dq\,dx - 2q\tan y\,dx\,dy - \sec^4 y\,dp\,dy\right) - s\left(dx^2 - \sec^4 y\,dy^2\right) = 0$$

Monge's subsidiary equations are

$$dq\,dx - 2q\tan y\,dx\,dy - \sec^4 y\,dp\,dy = 0 \quad \text{——} (1)$$

$$dx^2 - \sec^4 y\,dy^2 = 0 \quad \text{——} (2)$$

From eqn (2) we get following

$$dx - \sec^2 y\,dy = 0 \quad \text{——} (3)$$

$$dx + \sec^2 y\,dy = 0 \quad \text{——} (4)$$

Using eqns (3) & (1)

$$dq \cdot \sec^2 y\,dy - 2q\tan y\sec^2 y\,dy^2 + \sec^4 y\,dp\,dy = 0$$

$$\Rightarrow dq - 2q\tan y\,dy - \sec^2 y\,dp = 0$$

Multiplying $-\cos^2 y$ to each term

$$dp + 2q\sin y\cos y\,dy - \cos^2 y\,dq = 0$$

$$\Rightarrow dp - d(q\cos^2 y) = 0$$

On integrating

$$p - q\cos^2 y = a_1 \quad \text{——} (5)$$

Also from (3)

$$x - \tan y = b_1 \quad \text{------ (6)}$$

We get intermediate integral as

$$p - q\cos^2 y = f_1(x - \tan y) \quad \text{------ (7)}$$

Using eqns (4) & (1)

$$-dq\cdot\sec^2 y\,dy + 2q\tan y\sec^2 y\,dy^2 - \sec^4 y\,dp\,dy = 0$$

Dividing by $-\sec^2 y\,dy$

$$dq - 2q\tan y\,dy + \sec^2 y\,dp = 0$$

Multiplying $\cos^2 y$ to each term

$$dp - 2q\sin y\cos y\,dy + \cos^2 y\,dq = 0$$

$$\Rightarrow dp + d(q\cos^2 y) = 0$$

On integrating

$$p + q\cos^2 y = a_2 \quad \text{------ (8)}$$

Also from (4)

$$x + \tan y = b_2 \quad \text{------ (9)}$$

We get second intermediate equation as

$$p + q\cos^2 y = f_2(x + \tan y) \quad \text{------ (10)}$$

Solving (7) & (10)

$$p = \frac{1}{2}\left\{f_1(x - \tan y) + f_2(x + \tan y)\right\}$$

$$q = \frac{1}{2\cos^2 y}\left\{f_2(x + \tan y) - f_1(x - \tan y)\right\}$$

Now $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{1}{2}\left[\left\{f_1(x-\tan y) + f_2(x+\tan y)\right\}dx + \frac{1}{2\cos^2 y}\left\{f_2(x+\tan y) - f_1(x-\tan y)\right\}\right]$$

$$\Rightarrow dz = \frac{1}{2}\left[f_1(x-\tan y)\cdot\left(dx - \sec^2 y\,dy\right) + f_2(x+\tan y)\left(dx + \sec^2 y\,dy\right)\right]$$

$$\Rightarrow dz = \frac{1}{2}\left[f_1(x-\tan y)\,d(x-\tan y) + f_2(x+\tan y)\,d(x+\tan y)\right]$$

On integrating

$$\therefore z = \frac{1}{2}\left[\phi_1(x-\tan y) + \phi_2(x+\tan y)\right]$$

$$\therefore z = F_1(x-\tan y) + F_2(x+\tan y)$$ Ans

Ex Solve $pt - qs = q^3$.

Soln: Given $pt - qs = q^3$.

Putting $t = \frac{dq - s\,dx}{dy}$ we get

$$p\left(\frac{dq - s\,dx}{dy}\right) - qs = q^3$$

$$\left(p\,dq - q^3 dy\right) - s\left(p\,dx + q\,dy\right) = 0$$

So Monge's subsidiary equations are

$$p\,dq - q^3 dy = 0 \quad \text{---} (1)$$

$$\& \quad p\,dx + q\,dy = 0 \quad \text{---} (2)$$

From (2) $\quad p\,dx + q\,dy = 0$

$$\Rightarrow \quad dz = 0 \qquad (\because dz = p\,dx + q\,dy)$$

$$\Rightarrow \quad z = a_1 \quad \text{---} (3)$$

From (1) $\quad p\,dq - q^3 dy = 0$

$$\Rightarrow \quad p\,dq - q^2(-p\,dx) = 0 \qquad (\text{From } 2)$$

$$\Rightarrow \quad dq + q^2 dx = 0$$

$$\Rightarrow \quad \frac{dq}{q^2} + dx = 0$$

$$\Rightarrow \quad -\frac{1}{q} + x = b_1 \quad \text{---} (4)$$

From (3) & (4) we get intermediate eqn.

$$-\frac{1}{q} + x = f(z)$$

$$\Rightarrow \quad -\frac{\partial y}{\partial z} + x = f(z)$$

$$\Rightarrow \quad \frac{\partial y}{\partial z} - x = -f(z).$$

Integrating w.r to z

$$y - xz = -\int f(z)\,dz + C.$$

$$\Rightarrow \quad y = xz + f_1(z) + f_2(x) \quad (\because x \text{ is const})$$

Ans