and so $\alpha(t)=0$ identically

Proof of the Main theorem :-

If possible, let $\alpha_1(t)$ and $\alpha_2(t)$ be two determining functions corresponding to the same generating functions $f(s)$.

i.e. $f(s) = \int_0^\infty e^{-st}\, d\alpha_1(t)$

and $f(s) = \int_0^\infty e^{-st}\, d\alpha_2(t)$

$$\Rightarrow \int_0^\infty e^{-st}\, d\alpha_1(t) = \int_0^\infty e^{-st}\, d\alpha_2(t)$$

$$\Rightarrow \int_0^\infty e^{-st}\, [d\alpha_1(t) - d\alpha_2(t)] = 0$$

$$\Rightarrow \int_0^\infty e^{-st}\, d[\alpha_1(t) - \alpha_2(t)] = 0$$

{where $\alpha_1(t)$, $\alpha_2(t)$ are normalised
$\Rightarrow \alpha_1(t) - \alpha_2(t)$ is normalised}

$$\Rightarrow \alpha_1(t) - \alpha_2(t) = 0$$

$$\Rightarrow \alpha_1(t) = \alpha_2(t)$$

i.e. the determining functions is unique

———x———x———

Lemma I: If $\alpha(t)$ is of b.v. in $0 \le t \le \delta$, $\delta > 0$, then $\lim\limits_{T\to\infty} \frac{1}{\pi} \int_0^\delta \alpha(t) \frac{\sin Tt}{t}\, dt = \frac{\alpha(0+)}{2}$ [R.U-83]

Proof :- Let us consider

$$\lim_{T\to\infty} \frac{1}{\pi} \int_0^\delta \frac{\sin Tt}{t}\, dt = \lim_{T\to\infty} \frac{1}{\pi} \int_0^{T\delta} \frac{T \sin x}{x} \frac{dx}{T}$$

Putting $Tt = x \Rightarrow dt = \frac{dx}{T}$

also, $t = \delta \Rightarrow x = T\delta$ & $t = 0 \Rightarrow x = 0$

$$= \lim_{T\to\infty} \frac{1}{\pi} \int_0^{T\delta} \frac{\sin x}{x}\, dx$$

$$= \frac{1}{\pi} \int_0^\infty \left(\frac{\sin x}{x}\right) dx$$

$$= \frac{1}{\pi} \times \frac{\pi}{2} \quad \left\{ \because \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \right\}$$

$$= \frac{1}{2}$$

$$\therefore \lim_{T \to \infty} \frac{1}{\pi} \int_0^\delta \alpha(0+) \frac{\sin Tt}{t}\,dt = \frac{\alpha(0+)}{2}$$

There will be no loss of generality, if we assume that $\alpha(0+) = 0$

Now, $\alpha(0+) = 0$

$$\Rightarrow \lim_{h \to 0} \alpha(0+h) = 0$$

$$\Rightarrow \lim_{t \to 0+} \alpha(t) = 0$$

$$\Rightarrow |\alpha(t)| < \epsilon \quad \text{for } 0 < t \le \eta, \text{ where } \epsilon > 0, \eta > 0$$

$$\therefore I = \frac{1}{\pi} \int_0^\delta \alpha(t) \frac{\sin Tt}{t}\,dt = \frac{1}{\pi} \int_0^\eta \alpha(t) \frac{\sin Tt}{t}\,dt + \frac{1}{\pi} \int_\eta^\delta \alpha(t) \frac{\sin Tt}{t}\,dt$$

$$= \frac{1}{\pi} \alpha(0) \int_0^\xi \frac{\sin Tt}{t}\,dt + \frac{1}{\pi} \alpha(\eta) \int_\xi^\eta \frac{\sin Tt}{t} + \frac{1}{\pi} \int_\eta^\infty \alpha(t) \frac{\sin Tt}{t}\,dt$$

(Applying second Mean Value theorem for 1st integ

$$\therefore I = \frac{\alpha(\eta)}{\pi} \int_\xi^\eta \frac{\sin Tt}{t}\,dt + \frac{1}{\pi} \int_\eta^\delta \alpha(t) \frac{\sin Tt}{t}\,dt \quad \{\because \alpha$$

$$\Rightarrow |I| = \left| \frac{\alpha(\eta)}{\pi} \int_\xi^\eta \frac{\sin Tt}{t}\,dt + \frac{1}{\pi} \int_\eta^\delta \alpha(t) \frac{\sin Tt}{t}\,dt \right|$$

$$\le \frac{1}{\pi} |\alpha(\eta)| \cdot \left| \int_\xi^\eta \frac{\sin Tt}{t}\,dt \right| + \frac{1}{\pi} \left| \int_\eta^\delta \alpha(t) \frac{\sin Tt}{t} \right.$$

As $\left| \int_\xi^\eta \frac{\sin Tt}{t}\,dt \right| = \left| \int_{T\xi}^{T\eta} \frac{\sin x}{x}\,dx \right| < \left| \int_0^\infty \frac{\sin x}{x} \right.$

Putting $Tt = x$

and $\left|\int_{\eta}^{\infty} \alpha(t) \frac{\sin Tt}{t} dt\right| < \frac{\epsilon}{2}$ {by R-L theorem

which states that. If $f(x)$ is integrable on $[a, b]$, then $\int_a^b f(x) \sin nx \, dx \to 0$ as $n \to \infty$ & here $\frac{\alpha(t)}{t}$ is of b.v. so it is integrable in $0 < t \leq \delta$ i.e. $\eta \leq t \leq \delta$}

$$\therefore |g| \leq \frac{1}{\pi} \epsilon \frac{\pi}{2} + \frac{1}{\pi} \times \frac{\epsilon}{2}$$

$$\Rightarrow |g| = \frac{\epsilon}{2}\left[\frac{1}{\pi} + 1\right] = \epsilon_1 \text{ (say)}$$

$$\Rightarrow |g| < \epsilon_1$$

$$\Rightarrow \lim_{T \to \infty} g = 0 \Rightarrow \lim_{T \to \infty} \frac{1}{\pi} \int_0^{\delta} \alpha(t) \frac{\sin Tt}{t} dt = \frac{\alpha(0+)}{2}$$

---x---x---

Lemma 2:- If $\varphi(u)$ belongs to $L(-\infty, \infty)$ and is of b.v. in some two sided nbhd. of a point t then [R.U-82]

$$\lim_{T \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(u) \frac{\sin T(t-u)}{(t-u)} du = \frac{\varphi(t-) + \varphi(t+)}{2}$$

Proof :-

Let $$g = \lim_{T \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(u) \frac{\sin T(t-u)}{(t-u)} du$$

$$g = \lim_{T \to \infty} \frac{1}{\pi} \int_{-\infty}^{-R} \varphi(u) \frac{\sin T(t-u)}{(t-u)} du + \lim_{T \to \infty} \frac{1}{\pi} \int_{-R}^{R} \varphi(u) \frac{\sin T(t-u)}{(t-u)} du$$

$$+ \lim_{T \to \infty} \frac{1}{\pi} \int_{R}^{\infty} \varphi(u) \frac{\sin T(t-u)}{(t-u)} du$$

$$g = g_1 + g_2 + g_3$$

Given $\varphi(u) \in L(-\infty, \infty)$

$$\Rightarrow \frac{\varphi(u)}{(t-u)} \in L(-\infty, \infty)$$

$$\Rightarrow \frac{\varphi(u)}{t-u} \in L(-\infty, -R) \text{ & also } \in L(R, \infty)$$

$\therefore$ By R-L Theorem, $I_1 = I_3 = 0$

Now,

$$I = I_2 = \lim_{T\to\infty} \frac{1}{\pi}\int_{-R}^{R} \varphi(u) \frac{\sin T(t-u)}{(t-u)} du$$

$$= \lim_{T\to\infty} \frac{1}{\pi}\int_{-R}^{t} \varphi(u) \frac{\sin T(t-u)}{(t-u)} du$$

$$+ \lim_{T\to\infty} \frac{1}{\pi}\int_{t}^{R} \varphi(u) \frac{\sin T(t-u)}{(t-u)} du$$

For the 1st integral, put $(t-u) = y \Rightarrow u = (t-y)$

$\Rightarrow du = -dy$

when $u = t \Rightarrow y = 0$ & when $u = -R \Rightarrow y = (t+R)$

For the 2nd integral, put $(u-t) = z$ so that

$u = (z+t) \Rightarrow du = dz$

when $u = t$, $z = 0$ & $u = R$, $z = R-t$

$$\therefore I = I_2 = \lim_{T\to\infty} \frac{1}{\pi}\int_0^{R+t} \varphi(t-y) \frac{\sin Ty}{y} dy + \lim_{T\to\infty} \frac{1}{\pi}\int_0^{R-t} \varphi(t \ldots$$

$$= \frac{\varphi(t-0+)}{2} + \frac{\varphi(t+0+)}{2} \quad [\text{by lemma} \ldots$$

$$= \frac{\varphi(t-)}{2} + \frac{\varphi(t+)}{2}$$

$$I = I_2 = \frac{\varphi(t-) + \varphi(t+)}{2} \qquad \text{Proved}$$

Theorem :- State and prove complex inversion for... for Laplace integral

Statement :- If the Laplace-integral $f(s) = \int_0^\infty e^{-s\ldots}$ is cgt. absolutely on the real line $\sigma = \ldots$ then for $\varphi(u) \in \mathcal{L}(0,R)$, R being a... quantity, then

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{C-iT}^{C+iT} f(s)\, e^{st}\, ds \ldots$$

and for $\varphi(t)$ being of b.V. in the nbd. of that t
then
$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s)e^{st}\,ds = \frac{\varphi(t-)+\varphi(t+)}{2}, \; t>0$$
$$= \frac{\varphi(0+)}{2}, \text{ when } t=0$$

Proof:- To prove this theorem we first states the two Lemmas as below.

Lemma (1) If $\alpha(t)$ is b.V. in $0 \le t \le \delta$, $\delta>0$ then
$$\lim_{T\to\infty} \frac{1}{\pi}\int_0^{\delta} \alpha(t)\frac{\sin Tt}{t}\,dt = \frac{\alpha(0+)}{2}$$

Proof:-

Lemma 2: If $\varphi(u)$ belongs to $L(-\infty,\infty)$ and is of b
in some two sided nbhd. of a point t
$$\lim_{T\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(u)\frac{\sin T(t-u)}{(t-u)}\,du = \frac{\varphi(t-)+\varphi(t}{2}$$

of :-

Part of theorem:-

$$\text{Let } I = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s)\, e^{st}\, ds.$$

$$= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} e^{st}\left[\int_0^{\infty} e^{-su}\varphi(u)\,du\right]ds.$$

$$= \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\int_0^{\infty} e^{s(t-u)}\varphi(u)\, ds\, du$$

$$= \frac{1}{2\pi i}\int_0^{\infty}\varphi(u)\left[\int_{c-iT}^{c+iT} e^{s(t-u)}\, ds\right]du$$

$$= \frac{1}{2\pi i}\int_0^{\infty}\varphi(u)\left[\frac{e^{s(t-u)}}{t-u}\right]_{c-iT}^{c+iT} du$$

$$= \frac{1}{2\pi i}\int_0^\infty \varphi(u)\left[\frac{e^{s(t-u)}}{(t-u)}\right]_{c-iT}^{c+iT} du$$

$$= \frac{1}{2\pi i}\int_0^\infty \varphi(u)\,\frac{e^{(c+iT)(t-u)} - e^{(c-iT)(t-u)}}{t-u}\,du$$

$$\frac{1}{2\pi i}\int_0^\infty \varphi(u)\,\frac{e^{c(t-u)}}{(t-u)}\left[e^{iT(t-u)} - e^{-iT(t-u)}\right]du$$

$$\frac{1}{2\pi i}\int_0^\infty \varphi(u)\,\frac{e^{c(t-u)}}{(t-u)}\,2i\sin T(t-u)\,du$$

$$\frac{1}{\pi}\int_0^\infty \varphi(u)\,e^{c(t-u)}\,\frac{\sin T(t-u)}{(t-u)}\,du$$

$$\frac{1}{\pi}\int_{-\infty}^0 \varphi(u)\,e^{c(t-u)}\,\frac{\sin T(t-u)}{(t-u)}\,du + \frac{1}{\pi}\int_0^\infty \varphi(u)\,e^{c(t-u)}\,\frac{\sin T(t-u)}{(t-u)}\,du$$

if $\varphi(u) = 0$ for $u<0$

$$= \frac{1}{\pi}\int_{-\infty}^\infty \varphi(u)\,e^{c(t-u)}\,\frac{\sin T(t-u)}{(t-u)}\,du \quad \text{if } \varphi(u)=0, \text{ if } u<0.$$

$$\lim_{T\to\infty} I = \lim_{T\to\infty}\frac{1}{\pi}\int_{-\infty}^\infty \varphi(u)\,e^{c(t-u)}\,\frac{\sin T(t-u)}{(t-u)}\,du$$

when $\varphi(u)=0$ for $u<0$:

$$= \frac{\varphi(t+)\,e^{c(t-t+)} + \varphi(t-)\,e^{c(t-t-)}}{2} \quad \text{[by Lemma 2]}$$

$$= \frac{\varphi(t+)+\varphi(t-)}{2} \quad \text{when } t>0$$

$t=0$, $\varphi(t+) = \varphi(0+)$

$\varphi(t-) = \varphi(0-) = 0$

($\because$ we have taken $\varphi(u)=0$, when $u<0$)

$$\therefore \lim_{T\to\infty} I = \frac{\varphi(0+)+0}{2}, \quad t=0$$

$$= \frac{\varphi(0+)}{2} \quad \text{when } t=0$$