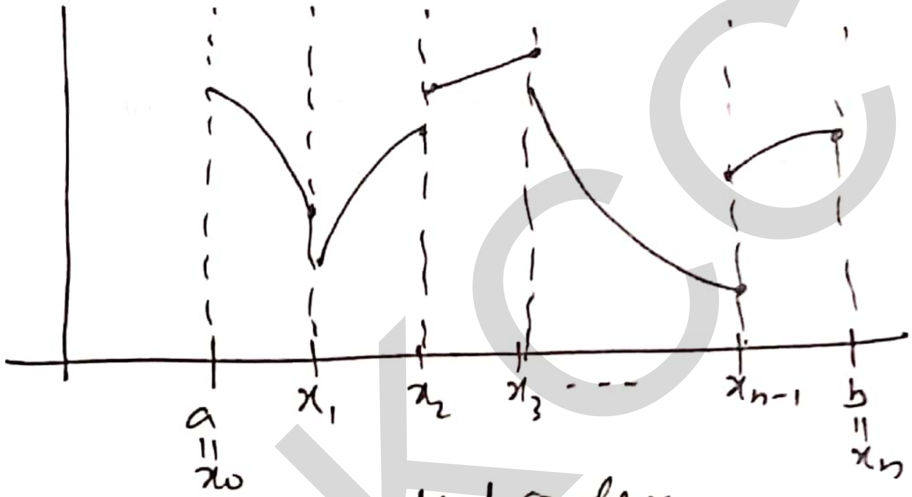
Integral Transform:

Piece-Wise or sectional continuity

A function $f(x)$ defined on $[a,b]$ is said to be piece-wise continuous if it is continuous & has finite left hand & right hand limits in every subinterval of $[a,b]$



Function of exponential order

A function $f(x)$ is said to be of exponential order 'a' as $x \to \infty$ if

$$\lim_{x\to\infty} e^{-ax} f(x) = \text{finite quantity.}$$

ie if for a given positive integer n_0, there exists a real number $M > 0$ such that

$$|e^{-ax} f(x)| < M \quad \forall \; x \geq n_0.$$

or

$$|f(x)| < Me^{ax} \quad \forall \; x \geq n_0.$$

Properties **Linear Property.**

Th^m Suppose $f_1(s)$ & $f_2(s)$ are Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively

$$\text{ie, } L\{F_1(t)\} = f_1(s) \quad \& \quad L\{F_2(t)\} = f_2(s)$$

then $L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$

Proof Given $f_1(s) = L\{F_1(t)\}$

$$\Rightarrow f_1(s) = \int_0^\infty e^{-st} F_1(t)\, dt.$$

Illy $\quad f_2(s) = \int_0^\infty e^{-st} F_2(t)\, dt$

Now $L\{c_1 F_1(t) + c_2 F_2(t)\}$

$$= \int_0^\infty e^{-st} \{c_1 F_1(t) + c_2 F_2(t)\}$$

$$= c_1 \int_0^\infty e^{-st} F_1(t)\, dt + c_2 \int_0^\infty e^{-st} F_2(t)\, dt$$

$$= c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$$

$$= c_1 f_1(s) + c_2 f_2(s)$$

Function of class A: A function $f(t)$ is said to be of class A if (i) It is piecewise continuous over every finite interval in the range $t \geq 0$. &
(ii) $f(t)$ is of exponential order as $t \to \infty$.

Integral transform:

An improper integral of the form

$$\int_{-\infty}^{\infty} K(s,t) \cdot F(t)\, dt \quad \text{———} \; (1)$$

is called integral transform of $F(t)$ if it is convergent. $K(s,t)$ is called kernel of transform.

Laplace Transform: If we take

$$K(s,t) = \begin{cases} e^{-st} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

then the integral transform (1) becomes

$$\int_{0}^{\infty} e^{-st} \cdot F(t)\, dt.$$

It is denoted by $f(s)$ or $L\{F(t)\}$

It is called Laplace transform of $\{F(t)\}$

So

$$f(s) = L\{F(t)\} = \int_{0}^{\infty} e^{-st} \cdot F(t)\, dt.$$

Th^m **First shifting theorem:**

If $L\{F(t)\} = f(s)$, then $L\{e^{at}F(t)\} = f(s-a)$.

Proof

Given $f(s) = L\{F(t)\}$

$$\Rightarrow f(s) = \int_0^\infty e^{-st} \cdot F(t)\, dt. \quad \text{---} (1)$$

Now

$$L\{e^{at}F(t)\} = \int_0^\infty e^{-st} \cdot e^{at} \cdot F(t)\, dt$$

$$= \int_0^\infty e^{-st+at} \cdot F(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot F(t)\, dt.$$

Let $s-a = u$.

$$= \int_0^\infty e^{-ut} \cdot F(t) \cdot dt$$

$$= f(u) \quad \text{(Comparing with (1))}$$

$$= f(s-a)$$

Proved

Second shifting th^m.

If $L\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$

then $L\{G(t)\} = e^{-as} f(s)$.

or $L\{F(t-a) \cdot H(t-a)\} = e^{-as} f(s)$

Where H is Heaviside unit function. $H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$

Proof

Given $L\{F(t)\} = f(s)$. &

$$G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$$

$$L\{G(t)\} = \int_0^\infty e^{-st}\, G(t)\, dt.$$

$$= \int_0^a e^{-st} G(t)\, dt + \int_a^\infty e^{-st} \cdot G(t)\, dt.$$

$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} G(t)\, dt$$

$$= 0 + \int_a^\infty e^{-st} F(t-a)\, dt$$

Let $t - a = p$

$dt = dp$

Also when $t = a$ $\quad p = 0$

$t = \infty$ $\quad p = \infty$

$$= \int_0^\infty e^{-s(p+a)} \cdot F(p)\, dp.$$

$$= e^{-sa} \int_0^\infty e^{-sp} \cdot F(p)\, dp.$$

$$= e^{-sa} \int_0^\infty e^{-st} \cdot F(t) \cdot dt$$

$$= e^{-sa} f(s).$$ Proved