Fourier transform (18)

Application of Infinite Fourier transform

Problem ① Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with

Conditions (i) $u(0,t) = 0$ (ii) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}, t = 0$

(iii) $u(x,t)$ is bounded.

Answer: Since $(u)_{x=0}$ is given and so we shall apply sine transform to the given equation, so that

$$\int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

$$\Rightarrow \frac{\partial}{\partial t} \int_0^\infty u \sin sx \, dx = \left[\sin sx \frac{\partial u}{\partial x}\right]_0^\infty - \int_0^\infty s \cos sx \frac{\partial u}{\partial x} dx$$

$$\Rightarrow \frac{d}{dt}\bar{u}_s = 0 - s\int_0^\infty \frac{\partial u}{\partial x} \cos sx \, dx, \text{ for } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty$$

Here $\bar{u}_s(s,t)$ is the Fourier sine transform of $u(x,t)$.

$$= -s\left[\cos sx \cdot u + \int s \sin sx \, u \, dx\right]_0^\infty$$

$$= -s\left[u \cos sx\right]_0^\infty - s^2 \int_0^\infty u \sin sx \, dx$$

$$= s\,u(0,t) - s^2 \bar{u}_s \text{ for } u \to 0 \text{ as } x \to \infty$$

$$= -s^2 \bar{u}_s, \text{ by boundry condition (i)}$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} + s^2 \bar{u}_s = 0$$

The solution of this $\bar{u}_s = A e^{-s^2 t}$ —— ①

S. Hasan
K.C.C.

Scanned with CamScanner

$\Rightarrow \overline{u_s}(s,0) = A$

$$\Rightarrow A = \int_0^\infty u(x,0)\sin sx \, dx$$

$$\Rightarrow A = \int_0^1 1 . \sin sx \, dx + \int_1^\infty 0 . \sin sx \, dx$$

$$= -\left[\frac{\cos sx}{s}\right]_{x=0}^{1} + 0 = -\left(\frac{\cos s - 1}{s}\right)$$

$$= \frac{1-\cos s}{s}$$

Now by ①, $\overline{u_s} = \left(\frac{1-\cos s}{s}\right) e^{-s^2 t}$

Applying inverse fourier transform

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos s}{s}\right) e^{-s^2 t} . \sin sx \, ds .$$

Ans.

Problem ② use finite cosine transform, to solve

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}, \quad 0 < x < \pi, \ t > 0$$

with the boundary conditions $\frac{\partial v}{\partial x} = 0$ when $x = 0$ and $x = \pi$, $t > 0$ and the initial condition $v = f(x)$, when $t = 0$, $0 < x < \pi$.

Answer Taking finite cosine transform of the given equation, $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$, we have

$$\frac{1}{k}\int_0^\pi \frac{\partial v}{\partial t}\cos px \, dx = \int_0^\pi \frac{\partial^2 v}{\partial x^2}\cos px \, dx$$

$$\Rightarrow \frac{1}{k}\frac{d\overline{v_c}}{dt} = \left[\cos px \frac{\partial v}{\partial x}\right]_0^\pi + \int_0^\pi p\sin px \frac{\partial v}{\partial x} \, dx$$

Scanned with CamScanner

$$\Rightarrow \frac{1}{k}\frac{d\overline{V_c}}{dt} = 0 + p\left[v(x,t)\sin px - p\int v\cos px\,dx\right]_{x=0}^{\pi}$$

$$= p\left[v(x,t)\sin px\right]_0^{\pi} - p\int_0^{\pi} v\cos px\,dx$$

$$= 0 - p^2\overline{V_c}$$

$$\Rightarrow \frac{d\overline{V_c}}{dt} = -kp^2\overline{V_c} \Rightarrow \frac{d\overline{V_c}}{dt} + kp^2\overline{V_c} = 0$$

Its solution is

$$V_c(p,t) = A e^{-p^2kt} \quad \text{—— (1)}$$

Now $V = f(x) \Rightarrow V_c(p,0) = \int_0^{\pi} f(x)\cos px\,dx$

Now by (1) $V_c(p,0) = A$

$$\therefore A = \int_0^{\pi} f(x)\cos px\,dx = A(p) \quad \text{—— (2)}$$

By definition of inverse Cosine finite Fourier transform, $v(x,t) = \frac{1}{\pi}\overline{V_c}(0,t) + \frac{2}{\pi}\sum_{p=1}^{\infty}\overline{V_c}\cos px$

$$\Rightarrow v(x,t) = \frac{A}{\pi} + \frac{2}{\pi}\sum_{p=1}^{\infty} A(p)\,e^{-p^2kt}\cos px,$$

where A is given by (2).

—— x ——

S. Hasan

K.C.C.